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On reflected interactions in elastic solids containing inhomogeneities

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ABSTRACT

Interactions in linear elastic solids containing inhomogeneities are examined using integral equations. Direct and reflected interactions are identified. Direct interactions occur simply because elastic fields emitted by inhomogeneities affect each other. Reflected interactions occur because elastic fields emitted by inhomogeneities are reflected by the specimen boundary back to the individual inhomogeneities. It is shown that the reflected interactions are of critical importance to analysis of representative volume elements. Further, the reflected interactions are expressed in simple terms, so that one can obtain explicit approximate expressions for the effective stiffness tensor for linear elastic solids containing ellipsoidal and non-ellipsoidal inhomogeneities. For ellipsoidal inhomogeneities, the new approximation is closely related to that of Mori and Tanaka. In general, the new approximation can be used to recover Ponte Castañeda–Willis' and Kanaun–Levin's approximations. Connections with Maxwell's approximation are established.

1. Introduction

The problem of calculating the effective properties of linear elastic solids containing inhomogeneities has a long history, which is well documented in recent books (Milton, 2002; Torquato, 2002). Currently, this problem can be successfully attacked using direct numerical simulations. Nevertheless there remains a considerable interest in developing accurate and easy-to-use approximations.

The most basic approximations for the effective properties are asymptotic solutions for dilute populations of inhomogeneities. In those, so-called dilute approximations, elastic interactions among inhomogeneities are neglected, simply because inhomogeneities are far apart from each other. Dilute approximations involve linear relationships between the effective stiffness (or compliance) tensor versus the volume fraction(s) of inhomogeneities. Perhaps the best known dilute approximation is for the effective shear modulus μ_d of an incompressible matrix containing randomly distributed rigid spherical inhomogeneities (Einstein, 1906):

$$\mu_d = \mu_0 \left(1 + \frac{5}{2} c\right), \tag{1}$$

where μ_0 is the shear modulus of the matrix and c is the volume fraction occupied by the spheres.

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Eshelby (1957) paved the way for a large body of work concerned with approximations applicable to composites with moderate and large volume fractions of inhomogeneities, in which elastic interactions among inhomogeneities have to be taken into account. Invariably those approximations are restricted to ellipsoidal inhomogeneities and involve nonlinear relationships between the effective stiffness (or compliance) tensor versus the volume fraction. This trend was recently challenged by Kachanov and Sevostianov (2013), who contend that it is often the case that the effective properties are more sensitive to geometric details of the individual inhomogeneities rather than to elastic interactions among inhomogeneities. Accordingly, Kachanov and Sevostianov advocate the need for approximations involving non-interacting non-ellipsoidal inhomogeneities.

Kachanov and Sevostianov (2013) also emphasize that non-interacting inhomogeneities do not have to be well separated, and therefore the corresponding approximations do not have to coincide with dilute approximations. To this end, let us consider two alternatives to (1), both based on the assumption that inhomogeneities do not interact:

$$\mu_{\mathrm{E}} = \mu_0 \frac{2}{2-5c} \approx \mu_0 \left(1 + \frac{5}{2}c + \frac{25}{4}c^2 + \cdots\right). \tag{2}$$

and

$$\mu_{\mathrm{M}} = \mu_0 \frac{2+3c}{2-2c} \approx \mu_0 \left(1 + \frac{5}{2}c + \frac{5}{2}c^2 + \cdots\right). \tag{3}$$

The first approximation is due to Eshelby (1957), who placed a reference inhomogeneity in a uniform stress field, and calculated μ_{E} by equating the complementary energies of the actual and effective materials. The second approximation is due to Torquato (2002), who extended Maxwell's (1873) approximation for conducting composites to classical elasticity. In Maxwell's approximation, the actual and effective materials are related by equating a remote field, induced by a cluster of actual non-interacting inhomogeneities, versus that, induced by an effective inhomogeneity. Let us mention that (1) can be obtained following Eshelby's (1957) procedure, with the provisions that (i) the reference inhomogeneity is placed in a uniform strain rather than stress field, and (ii) the effective shear modulus is calculated by equating the strain rather than complementary energies.

Approximations (1)–(3) coincide to the first order in c, but their quadratic terms are significantly different. In choosing among (1)–(3), one can easily rule out (1), simply because μ_d is below the lower Hashin–Shtrikman bound, which happens to coincide with μ_{M}. The choice between μ_{E} and μ_{M} is less obvious, although one can argue that μ_{M} should be preferred since μ_{E} breaks down at $c=2/5$, which is well below the maximum volume fraction for randomly packed spheres. Also it is intriguing that μ_{M} coincides with μ_{MT} obtained from the approximation of Mori and Tanaka (1973), in which elastic interactions among inhomogeneities are presumably taken into account. Thus the concept of non-interacting inhomogeneities per se appears to be somewhat ambiguous and needs to be delineated.

In this paper, we identify two types of elastic interactions among inhomogeneities within representative volume elements (RVEs). First, we identify direct interactions, responsible for the effective field of each inhomogeneity being affected by its neighbors. Second, we recognize that inhomogeneities also interact with each other indirectly, because elastic fields emitted by inhomogeneities are reflected by the specimen boundary, and those reflections contribute to the effective fields of the individual inhomogeneities. We refer to these interactions as reflected. It may appear that the reflected interactions are insignificant in comparison to the direct ones, and their description is far too complicated. Perhaps the principal contribution of this paper is to demonstrate that in RVEs the reflected interaction are important and they are straightforward to quantify. Furthermore, by neglecting the direct but not reflected interactions, one obtains simple approximations for the effective properties of composites containing ellipsoidal and non-ellipsoidal inhomogeneities, which compare favorably with approximations well established in the literature.

In mathematical terms, our development is strongly influenced by O'Brien's (1979) work, who derived an integral equation applicable to both finite and infinite solids. In particular, that equation can be used for evaluating the so-called thermodynamic limit in which the specimen size and the number of inhomogeneities tend to infinity simultaneously. As a result, our approach has the following major advantages:

(i) It avoids convergence difficulties associated with transition from finite to infinite solids.
(ii) It is valid for any boundary conditions associated with prescribed macroscopic stresses, or strains, or their combinations.
(iii) It yields consistent effective stiffness ($\boldsymbol{L}$) and compliance ($\boldsymbol{M}$) tensors, so that $\boldsymbol{LM}=\boldsymbol{I}$, where $\boldsymbol{I}$ is the fourth rank symmetric identity tensor.
(iv) Approximate expressions for the effective stiffness tensor are transparent, as they are obtained by neglecting certain terms in the integral equation.

Following O'Brien (1979), our development is based on classical calculus. This allows us to explain better the physical significance of various terms associated with singular integral operators.

The remainder of this paper is organized as follows. In Section 2, we define the effective elastic properties for a specimen containing inhomogeneities. In Section 3, we develop governing integral equations for several model problems, which allow us to delineate the mathematical structure of the direct and reflected interactions. In Sections 4 and 5, by assuming that the