Journal of the Mechanics and Physics of Solids 61 (2013) 1712–1736

Contents lists available at SciVerse ScienceDirect

Journal of the Mechanics and Physics of Solids

journal homepage: www.elsevier.com/locate/jmps

Continuation of equilibria and stability of slender elastic rods using an asymptotic numerical method

A. Lazarus [a], J.T. Miller [b], P.M. Reis [a,b,*]

[a] Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
[b] Department of Civil and Environmental Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

ARTICLE INFO

Article history:
Received 19 December 2012
Received in revised form
28 February 2013
Accepted 4 April 2013
Available online 19 April 2013

Keywords:
Elastic rods
Quaternions
Path-following techniques
Equilibrium
Stability

ABSTRACT

We present a theoretical and numerical framework to compute bifurcations of equilibria and stability of slender elastic rods. The 3D kinematics of the rod is treated in a geometrically exact way by parameterizing the position of the centerline and making use of quaternions to represent the orientation of the material frame. The equilibrium equations and the stability of their solutions are derived from the mechanical energy which takes into account the contributions due to internal moments (bending and twist), external forces and torques. Our use of quaternions allows for the equilibrium equations to be written in a quadratic form and solved efficiently with an asymptotic numerical continuation method. This finite element perturbation method gives interactive access to semi-analytical equilibrium branches, in contrast with the individual solution points obtained from classical minimization or predictor–corrector techniques. By way of example, we apply our numerics to address the specific problem of a naturally curved and heavy rod under extreme twisting and perform a detailed comparison against our own precision model experiments of this system. Excellent quantitative agreement is found between experiments and simulations for the underlying 3D buckling instabilities and the characterization of the resulting complex configurations. We believe that our framework is a powerful alternative to other methods for the computation of nonlinear equilibrium 3D shapes of rods in practical scenarios.

1. Introduction

Filaments, rods and cables are encountered over a wide range of length-scales, both in nature and technology, providing outstanding kinematic freedom for practical applications. Given their slender geometry, they can undergo large deformations and exhibit complex mechanical behavior including buckling, snap-through and localization. A predictive understanding of the mechanics of thin rods has therefore long motivated a large body of theoretical and computational work, from Euler's elastica in 1744 (Levien, 2008) and Kirchhoff's kinetic analogy in 1859 (Dill, 1992) to the burgeoning of numerical approaches such as finite element-based methods in the late 20th century (Zienkiewicz et al., 2005), and the more recent algorithms based on discrete differential geometry (Bergou et al., 2008). Today, these advances in modeling of the mechanics of slender elastic rods are helping to tackle many cutting-edge research problems. To name just a few, these

* Corresponding author at: Department of Civil and Environmental Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA. Tel.: +1 6173243325.
E-mail addresses: alazarus@mit.edu (A. Lazarus), flymile@mit.edu (J.T. Miller), preis@mit.edu (P.M. Reis).

0022-5096/$ - see front matter
http://dx.doi.org/10.1016/j.jmps.2013.04.002

range from the supercoiling of DNA (Coleman and Swigon, 2000; Marko and Neukirch 2012), self-assembly of rod-coil block copolymers (Wang et al., 2012), design of nano-electromechanical resonators (Lazarus et al., 2010b, 2010a), development of stretchable electronics (Sun et al., 2006), computed animation of hairs (Bertails et al., 2006) and coiled tubing operations in the oil-gas industries (Wicks et al., 2008).

An ongoing challenge in addressing these various problems involves the capability to numerically capture their intrinsic geometric nonlinearities in a predictive and efficient way. These nonlinear kinematic effects arise from the large displacements and rotations of the slender structure, even if its material properties remain linear throughout the process (Audoly and Pomeau, 2010). As a slender elastic rod is progressively deformed, the nonlinearities of the underlying equilibrium equations become increasingly stronger leading to higher densities in the landscape of possible solutions for a particular set of control parameters. When multiple stable states coexist, classic step-by-step algorithms such as Newton–Raphson methods (Crisfield, 1991) or standard minimization techniques (Luenberger, 1973) are often inappropriate since, depending on the initial guess, they may not converge toward the desired solution, or any solution. Addressing these computational difficulties calls for alternative numerical techniques, such as well-known continuation methods (Riks, 1979; Koiter, 1970). Continuation techniques are based on coupling nonlinear algorithms (e.g. predictor–corrector (Riks, 1979) or perturbation methods (Koiter, 1970)) with an arc-length description to numerically follow the fixed points of the equilibrium equations as a function of a control parameter, that is often a mechanical or geometrical variable of the problem. With the goal of determining the complete bifurcation diagram of the system, these methods enable the computation of all of the equilibrium solution branches, as well as their local stability.

Two main approaches can be distinguished for continuing the numerical solutions of geometrically nonlinear problems. The first includes predictor–corrector methods whose principle is to follow the nonlinear solution branch in a stepwise manner, via a succession of linearizations and iterations to achieve equilibrium (Crisfield, 1991). These methods are now widely used, particularly for the numerical investigation of solutions of conservative dynamical systems, with the free path-following mathematical software AUTO being an archetypal example (Doedel, 1981). Quasi-static deformations of slender elastic rods have been intensively studied using this software (Thompson and Champneys, 1996; Furrer et al., 2000; Healey and Mehta, 2005), mostly due to the analogy between the rod's equilibrium equations with the spinning top's dynamic equations (Davies and Moon, 1993). Although popular and widely used, the main difficulty with these algorithms involves the determination of an appropriate arc-length step size, which is fixed *a priori* by the user, but may be intricately dependent on the system's nonlinearities along the bifurcation diagram. A smaller step size will favor the computation of the highly nonlinear part of the equilibrium branch, such as bifurcation points, but may also impractically increase the overall computational time. On the other hand, a larger step size may significantly compromise the accuracy and resolution of the results.

The second class of continuation algorithms, which have received less attention, is a perturbation technique called the *Asymptotic Numerical Method* (ANM), which was first introduced in the early 1990s (Damil and Potier-Ferry, 1990; Cochelin, 1994). The underlying principle is to follow a nonlinear solution branch by applying the ANM in a stepwise manner and represent the solution by a succession of local polynomial approximations. This numerical method is a combination of asymptotic expansions and finite element calculations which allows for the determination of an extended portion of a nonlinear branch at each step, by inverting a unique stiffness matrix. This continuation technique is significantly more efficient than classical predictor–corrector schemes. Moreover, by taking advantage of the analytical representation of the branch within each step, it is highly robust and can be made fully automatic. Unlike incremental-iterative techniques, the arc-length step size in ANM is adaptative since it is determined *a posteriori* by the algorithm. As a result, bifurcation diagrams can be naturally computed in an optimal number of iterations. The method has been successively applied to nonlinear elastic structures such as beams, plates and shells but the geometrical formulations were limited to the early post-buckling regime and to date, no stability analyses were performed with ANM (Cochelin et al., 1994; Zahrouni et al., 1999; Vannucci et al., 1998).

In this paper, we develop a novel implementation of the semi-analytical ANM algorithm to follow the equilibrium branches and local stability of slender elastic rods with a geometrically-exact 3D kinematics. In Section 2, we first describe the 3D kinematics where the rod is represented by the position of its centerline and a set of unit quaternions to represent the orientation of the material frame. In Section 3, we then derive the closed form of the rod's cubic nonlinear equilibrium equations. To do this, we minimize the geometrically-constrained mechanical energy including internal bending and twisting energy, as well as the work of external forces and moments. Introducing the flexural and torsional internal moments in the vector of unknowns yields differential equilibrium equations that are quadratic. In Section 4, we proceed by presenting the numerical method developed to compute the equilibrium solutions. Using a finite-difference scheme, the discretized system of equilibrium equations can be solved with the ANM algorithm, which is particularly efficient for computing our algebraic quadratic form. The local stability of the computed equilibrium branches is assessed by a second order condition on the constrained energy. Finally, we describe how to implement this numerical method in the open source software MANlab; a user-friendly, interactive and Matlab-based path-following and bifurcation analysis program (Arquier, 2007; Karkar et al., 2010). In Section 5, we develop our own precision model experiment for the fundamental problem of the writhing of a clamped, heavy and naturally curved elastic rod. Because the writhing configuration has been studied extensively in the past (Thompson and Champneys, 1996; Goriely and Tabor, 1998a; VanderHeijden and Thompson, 2000; Goyal et al., 2008), this is an ideal scenario in which to challenge our numerical model against experimental results. Our simulations are robust, computationally time-efficient and exhibit excellent quantitative agreements with our experiments, demonstrating the predictive power of our framework.