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A model for a constrained, finitely deforming, elastic solid with rotation gradient dependent strain energy, and its specialization to von Kármán plates and beams

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ABSTRACT

The aim of this paper is to develop the governing equations for a fully constrained finitely deforming hyperelastic Cosserat continuum where the directors are constrained to rotate with the body rotation. This is the generalization of small deformation couple stress theories and would be useful for developing mathematical models for an elastic material with embedded stiff short fibers or inclusions (e.g., materials with carbon nanotubes or nematic elastomers, cellular materials with oriented hard phases, open cell foams, and other similar materials), that account for certain longer range interactions. The theory is developed as a limiting case of a regular Cosserat elastic material where the directors are allowed to rotate freely by considering the case of a high "rotational mismatch energy". The theory is developed using the formalism of Lagrangian mechanics, with the static case being based on Castigliano's first theorem. By considering the stretch $\mathbf{U}$ and the rotation $\mathbf{R}$ as additional independent variables and using the polar decomposition theorem as an *additional constraint equation*, we obtain the governing and as well as the boundary conditions for finite deformations. The resulting equations are further specialized for plane strain and axisymmetric finite deformations, deformations of beams and plates with small strain and moderate rotation, and for small deformation theories. We also show that the boundary conditions for this theory involve "surface tension" like terms due to the higher gradients in the strain energy function. For beams and plates, the rotational gradient dependent strain energy does not require additional variables (unlike Cosserat theories) and additional differential equations; nor do they raise the order of the differential equations, thus allowing us to include a material length scale dependent response at no extra "computational cost" even for finite deformation beam/plate theories

1. Introduction

Classical elasticity has served us very well in the development of models for the response of materials used in a number of fields, ranging from structural mechanics to biomechanics. However, the advent of new multi-constituent materials such as nematic elastomers and carbon nanotube composites (see, e.g., Cadek et al., 2004) has created new challenges in modeling, especially when the spatial resolution (or length scale) is comparable to the size of the secondary constituents. Such a situation arises, for example, when one considers environment resistant coatings made of CNT reinforced materials

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(see Kaempgen et al., 2005; Chen et al., 2003). Recently, there has been a great deal of interest in the flexoelectric effect (see Catalan et al., 2011), which is a size dependent strain gradient effect on the polarization of ferroelectrics and even induces piezo-electric response in non-piezoelectric materials at very small scales. Modeling of these materials requires the consideration of very small length scales over which the neighboring secondary constituents interact. With regard to the specific constitutive model introduced here, namely elastic bodies with strain energy dependent upon rotation gradients, there have been several papers in the literature showing experimental evidence in porous solids (see Lakes, 1986) and in honeycomb structures (see Mora and Waas, 2007).

Theories of micro-structured media undergoing finite deformations have been developed dating back to the 1960s and have reached a state of maturity (see, e.g., Green et al., 1965a; Green and Naghdi, 1995). However, much of the work on modeling such interactions has occurred in the liquid phase, for example, the remarkably successful Frank–Leslie–Ericksen theory for liquid crystals (see Leslie, 1992).

When attention is turned to elastic solids, there is a very large and well established body of work on small deformation couple stress theories with constrained micro-rotation beginning with the early work of Mindlin (1963). There has also been a surge of interest in this area in recent times (see e.g., Lubarda and Markenscoff, 2000; Yang et al., 2002; Ma et al., 2008; Reddy, 2011) with a wealth of exact solutions and comparisons with experiments (see, e.g., Yang and Lakes, 1982). Furthermore, such models have been used in a wide variety of applications such as in periodic grid structures (including studies of buckling phenomena) (see Bazant, 1971; Bažant and Christensen, 1972) as well as in cellular solids (see Onck, 2002). However, these approaches are based on simply postulating the equations directly (or deriving them in the limit of small deformation of discrete grids and trusses) for small deformation with no attempt at a systematic derivation from finite deformation. In other words, there is no corresponding finite deformation couple stress theory *with constrained rotations*. We hasten to add that Cosserat and micropolar theories of media are well developed and have their own extensive literature (see e.g., Toupin, 1964; Green and Naghdi, 1967, 1995; Hård af Segerstad et al., 2009), but in the case of solids, most of the work has been in the area of the finite deformation of classical plates and shells (see, e.g., Green et al., 1965b) and crystal plasticity (see, e.g., Naghdi and Srinivasa, 1993, 1994; Le and Stumpf, 1996), as well as for couple stress theories with *unconstrained micro-rotations* (see, e.g., Hård af Segerstad et al., 2009). A theory of a finitely deforming three-dimensional elastic Cosserat continuum (which is the 3-D counterpart of Cosserat rods and shells) where the motion of the directors (or the micro-rotation tensor) *is fully constrained*, has not been developed. We seek to fill this need in this work. We also extend the classical moderate rotation plate and beam theories to include additional small length scale effects.

Rather than directly posting the balance laws of the Cosserat continuum, the elasticity of the Cosserat continuum considered here offers the possibility of using a Lagrangian mechanics-based approach for the development of the field equations. Such procedures for material response that depend upon higher gradients have the added advantage of enabling the identification of proper boundary conditions even when the response is not elastic (see e.g. Baek and Srinivasa, 2003b). However, the central difficulty in developing a theory of Cosserat continua with the directors constrained to rotate with the body is the fact that, *while there is an explicit representation for the rotation in terms of the curl of the displacement for small deformation, no such representation in terms of displacements is available for a general finite deformation in 3-D*. Thus, it is not possible to choose generalized coordinates that automatically enforce the required constraint on the rotation. In this paper, we overcome this difficulty by using the polar decomposition of the deformation gradient as a constraint equation and using Lagrange multipliers in developing the theory. Systematic linearization of the resulting equations then leads to the well known couple stress theory with constrained micro-rotations.

We also show that for two popular classes of problems with wide technological importance, namely, plane strain problems and axisymmetric problems, *there is an explicit representation of the rotation in terms of the displacement gradient* so that, for such problems, the constraint equations may be eliminated.

2. A three-dimensional hyperelastic Cosserat continuum

In order to set the stage for the developments that follow, we first present, in a brief manner, the well known theory of elastic Cosserat continua with unconstrained micro-rotations albeit from a Lagrangian mechanics point of view. Thus, consider a composite material composed of a relatively compliant matrix in which are embedded ellipsoidal stiff constituents. For example, in a nematic elastomer, the matrix is a soft rubber and the hard constituents are the nematic phase. In a CNT reinforced composite, the CNTs are considered the stiff phase. The presence of the stiff phase decreases the compliance of the composite, especially restricting bending modes due to the interference between the stiff constituents. It is presumed that such materials can be modeled by using the theory of Cosserat continua.

Consider a body $\mathbb{B}$ with particles X that occupy a reference position $\mathbf{X}$[1] at time $t=0$ and which subsequently occupies a position $\mathbf{x}$ at time t. The motion of the body is given by a mapping $\mathbf{x}=\chi(\mathbf{X},t)$ that maps positions at $t=0$ to those at t. As with other Cosserat continua, in order to account for the presence of a high concentration of stiff particles that have definite orientation, each particle is also endowed with an orientation tensor $\boldsymbol{\Theta}$, which is a proper orthogonal tensor. The

[1] In this paper it suffices to use the configuration at time $t=0$ as the reference configuration, although it is not necessary to do so.