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An RVE-based multiscale theory of solids with micro-scale inertia and body force effects

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ABSTRACT

A multiscale theory of solids based on the concept of representative volume element (RVE) and accounting for micro-scale inertia and body forces is proposed. A simple extension of the classical Hill–Mandel Principle together with suitable kinematical constraints on the micro-scale displacements provide the variational framework within which the theory is devised. In this context, the micro-scale equilibrium equation and the homogenisation relations among the relevant macro- and micro-scale quantities are rigorously derived by means of straightforward variational arguments. In particular, it is shown that only the fluctuations of micro-scale inertia and body forces about their RVE volume averages may affect the micro-scale equilibrium problem and the resulting homogenised stress. The volume average themselves are mechanically relevant only to the macro-scale.

1. Introduction

Classical multiscale theories to predict the mechanical behaviour of solids with a microstructure have their origins in the pioneering works of Hill (1963, 1965a,b, 1972), Hashin and Shtrikman (1963), Budiansky (1965) and Mandel (1971), among others. Over the last two decades or so, theories relying on the averaging of stresses and strains over a *representative volume element (RVE)* have become remarkably popular in the prediction of overall properties of heterogeneous solids in non-linear regimes. Their use in practical applications relies almost exclusively on techniques of computational homogenisation (Kouznetsova et al., 2004; Michel et al., 1999; Miehe et al., 1999; Terada and Kikuchi, 2001). These techniques have reached such a level of maturity that multiscale theories are now beginning to find their way in specialised applications with a very promising prospect of becoming a much needed tool to help the design of new materials and the prediction of constitutive behaviours resulting from the interaction of complex microstructural phenomena (Paggi and Wriggers, 2012; Saavedra Flores and Friswell, 2012).

Despite the success history of RVE-based multiscale theories, the consideration of inertia and body forces in general appears not to have been satisfactorily addressed to date. In the classical work of Hill (1972) inertia and body forces are not considered. In the more recent literature, body forces are often removed from the theory on the basis of questionable arguments. Inertia forces, in turn, have rarely been considered in this context. In the few reported attempts to incorporate inertia effects, the theory appears to be unclear and suffers from significant inconsistencies.

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At present, the increasing interest in so-called metamaterials – microstructured materials displaying useful exotic macroscopic behaviour – puts pressure on the development of robust multiscale theories capable of predicting the overall response by accounting for the interaction of (possibly complex) phenomena at the micro-scale (Del Vescovo and Giorgio, 2014). In this context, the consideration of inertia and body forces may become crucial. The macroscopic mechanical response of acoustic metamaterials, for example, is dictated by dynamic phenomena at the micro-scale. Any attempt to model such materials by means of RVE-based multiscale theories must properly address the consideration of micro-scale inertia effects.

Our purpose in the present paper is to show in a clear manner how inertia effects and body forces in general can be rigorously accounted for in such theories. To this end we cast the theory within a framework relying entirely on the two fundamental principles of *kinematical admissibility* and *Multiscale Virtual Power* – the latter expressed as a variational statement of an extended version of the Hill–Mandel Principle of Macrohomogeneity (Hill, 1972; Mandel, 1971). These provide the essential link between the macro- and micro-scale kinematics and virtual power, respectively. Within this framework, once the macro- and micro-scale kinematical variables are defined and appropriate kinematical constraints are postulated to link them in a consistent manner, *all* equations of the resulting multiscale theory – including RVE equilibrium and the homogenisation relations for force- and stress-like variables – are *derived* (rather than postulated) exclusively by means of straightforward variational arguments. Here we should point out that the recent literature provides examples where extended versions of the Hill–Mandel Principle have been used for this purpose, but a deeper look into the resulting models reveals significant inconsistencies. Such inconsistencies stem either from insufficient kinematical constraints being imposed to ensure a meaningful link between the macro- and micro-scale kinematics or from the variationally inconsistent manner in which kinematical constraints have been taken into account in the treatment of the corresponding model. We begin by introducing the proposed framework in Section 2, against the background provided by the well-known classical theory (in the absence of inertia and body forces). Our main result is presented in Section 3 where we extend these ideas to the case of non-zero inertia and body forces. In this context, the role of inertia and body forces naturally emerges very clearly, allowing one to easily see how they can be taken into account in a consistent manner. A discussion of our findings follows in Section 4 and the paper closes with some concluding remarks made in Section 5.

2. Classical theory. Review

Consider a solid continuum that occupies a region Ω of the three-dimensional Euclidean space in its reference configuration. A wide family of so-called multiscale constitutive theories are derived based on the idea that any point $\mathbf{x}$ of Ω is associated with a *representative volume element* (RVE), occupying a reference domain Ω_μ of characteristic length ℓ_μ much smaller than the characteristic length ℓ of Ω. The domains Ω and Ω_μ are referred to as the macro- and micro-scale, respectively.

Classical multiscale theories (de Souza Neto and Feijóo, 2006, 2008, 2010; Perić et al., 2011) that predict the macro-scale mechanical behaviour from the constitutive properties of the corresponding micro-scale can be entirely derived from two fundamental principles: (i) *kinematical admissibility*; and (ii) *Multiscale Virtual Power*, that govern the transition between the two scales. Although by no means absolutely necessary, the derivation of all equations of the theory as a consequence of these two principles alone provides, in our view, a robust framework to treat the problem. In particular, it allows extensions of the classical theory (such as the one that is the subject matter of the present paper) to be devised in very clear steps on solid theoretical grounds. We remark that this approach has been recently employed with success by Sánchez et al. (2013) in the derivation of a multiscale theory accounting for material failure associated with micro-scale strain localisation phenomena. We begin by illustrating in the following the use of this idea in the case of the classical theory, where inertia and body forces are assumed absent.

2.1. Kinematical homogenisation and kinematical admissibility

Let $\mathbf{y} \in \Omega_\mu$ denote the coordinates of an arbitrary point of the RVE associated with a point $\mathbf{x} \in \Omega$. Without loss of generality we shall assume the origin of the micro-scale coordinate system to be located at the centroid of Ω_μ, i.e.

$$\int_{\Omega_\mu} \mathbf{y}\, d\Omega_\mu = 0. \tag{1}$$

A fundamental assumption in the present class of theories is that the micro-scale displacement field $\mathbf{u}_\mu$ over Ω_μ can be expanded as

$$\begin{aligned}\mathbf{u}_\mu(\mathbf{y}) &= \mathbf{u}(\mathbf{x}) + \nabla\mathbf{u}(\mathbf{x})\,\mathbf{y} + \tilde{\mathbf{u}}_\mu(\mathbf{y}) \\ &= \mathbf{u}(\mathbf{x}) + [\mathbf{F}(\mathbf{x}) - \mathbf{I}]\,\mathbf{y} + \tilde{\mathbf{u}}_\mu(\mathbf{y}),\end{aligned} \tag{2}$$

where $\mathbf{u}(\mathbf{x})$ is the displacement of the corresponding point $\mathbf{x}$ of the macro-scale, $\nabla(\cdot)$ denotes the gradient of $(\cdot)$ with respect to the macro-scale coordinates,

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u} \tag{3}$$

is the macro-scale deformation gradient and

$$\tilde{\mathbf{u}}_\mu \equiv \mathbf{u}_\mu - \mathbf{u} - (\mathbf{F} - \mathbf{I})\mathbf{y} \tag{4}$$

is defined as the *displacement fluctuation field* of the RVE. In view of (2) and (3) the micro-scale deformation gradient field,

$$\mathbf{F}_\mu = \mathbf{I} + \nabla_\mu \mathbf{u}_\mu, \tag{5}$$

with ∇_μ denoting the gradient with respect to the micro-scale coordinates, is equivalently expressed as

$$\mathbf{F}_\mu(\mathbf{y}) = \mathbf{I} + \nabla\mathbf{u}(\mathbf{x}) + \nabla_\mu\tilde{\mathbf{u}}_\mu(\mathbf{y}) = \mathbf{F}(\mathbf{x}) + \nabla_\mu\tilde{\mathbf{u}}_\mu(\mathbf{y}). \tag{6}$$

That is, the micro-scale deformation gradient field is a sum of the macro-scale deformation gradient, inserted