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# Statics of loose triangular embankment under Nadai's sand hill analogy

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ABSTRACT

In structural mechanics, Nadai's sand hill analogy is the interpretation of an ultimate torque applied to a given structural member with a magnitude that is analogously twice the volume of stable sand heap which can be accommodated on a transverse cross-section basis. Nadai's analogy is accompanied by his observation of a loose triangular embankment, based on the fact that gravitating loose earth is stable if inclined just under the angle of repose. However, Nadai's analysis of stress distribution in a planar sand heap was found to be inaccurate because the total pressure obtained from Nadai's solution is greater than the self-weight calculated from the heap geometry. This raises a question about the validity of his observation in relation to the analogy. To confirm his criterion, this article presents and corrects the error found in Nadai's solution by analyzing a radially symmetric stress field for a wedge-shaped sand heap with the purpose of satisfying both force balance and Nadai's closure. The fundamental equation was obtained by letting the friction state vary as a function of angular position and deduce it under the constraint that the principal stress orientation obeys Nadai's closure. The theoretical solution sufficiently agreed with the past experimental measurements.



## 1. Introduction

Rapid development in the subject of plasticity appeared with the publication of the classical books of Arpad L. Nádai (Nadai, 1963). His works are considered as significant contributions to the field of engineering materials, particularly to the areas of structural mechanics and geo-mechanics. Nadai's sand hill or sand heap analogy is widely known as the graphical interpretation of a fully plastic condition progressing throughout a twisted member. The stress function solution can be visually depicted as a surface of sand heap which is poured upon a horizontal plate shaped in a cross-section. This analogy is based on the fact that a slope of dry sand has a natural angle of repose that is slightly greater than the angle of internal friction. A heap of sand tries to keep a constant slope everywhere because the sand particles upslope will cascade down

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once the slope exceeds this limit. Therefore, if sand is added slowly and continually, a heap will grow and reach a unique steady state, with the shape remaining unchanged with the appearance of the peak, edges or ridge line. Nadai observed that by replacing a frictional coefficient to ultimate shear stress, the ultimate torque applied to any given structural member is analogously twice the volume of stable sand heap growing on that cross-section. An outline of his analogy is elaborated in Figs. 1 and 2. A detailed explanation can be found in the first volume of his monograph and other relevant textbooks in structural mechanics and plasticity theory (Calladine, 2000; Kachanov, 1971; Richards, 2000).

Nadai's achievements in geo-mechanics began with the second volume of his monograph. It is evident that Nadai also tried to investigate stress distribution in a planar sand heap under plane-strain condition. In accordance with his sand hill analogy, Nadai pointed out that gravitating loose earth is stable if inclined at an angle just lower than the angle of repose. Any small lump of sand added to the surface of a heap will cause a flow confined only to the surface, so the deeper sand lying below the surface remains stable and immobilized. Let us regard his observation of sand heap as Nadai's sand heap criterion. Experiments on various topographies of steady sand heaps verifying Nadai's sand heap criterion can be found in Pauli and Gioia (2007). Some basic experiments carried out by the authors are also demonstrated in Figs. 3 and 4 to verify that the final shape of the sand heap is independent of the sand deposition method used.

To obtain the stress solution, Nadai considered a rigid-elastic body of a long sand heap whose symmetrical surfaces are stabilized just under the Coulomb failure criterion with zero cohesion. Nadai noticed that an infinitesimal interface of zero traction resisted by friction along the sliding plane can provide a major compressive direction in accordance with the Coulomb failure criterion of pure friction material, which is angled precisely at the middle of the slope of a sliding plane and the direction of gravity. Moreover, the condition of zero shear stress along the central plane clearly reveals that the major compressive direction under the ridge is parallel to the gravity direction. According to two known limits of direction of major compressive stress, the assumed linear relation between the angle of major compressive stress and the polar angle measured from the ridge was introduced. Finally, the states of stress satisfying the equilibrium configuration under self-weight loading were formulated. However, it was found later that his solution contains an error of inappropriately associating vertical stress directly under the ridge of the sand heap equal to the height of the heap times the bulk unit weight of sand. By virtue of partial support from frictional resistance among sand particles, it is understandable that the weight of a thin column of sand under the apex cannot be wholly transferred to the base. In fact, none of the experimental results in wedge-shaped sand heap confirm that the basal pressure in the middle of the heap represents a full geostatic

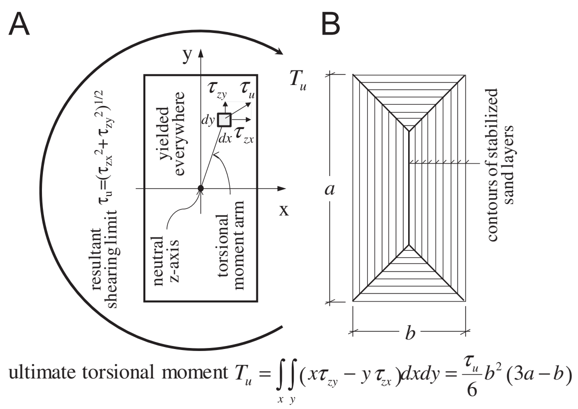


**Fig. 1.** (A) State of shear stress under fully plastic condition due to ultimate torsional moment and (B) topography of sand heap piled into a rectangular cross-section.

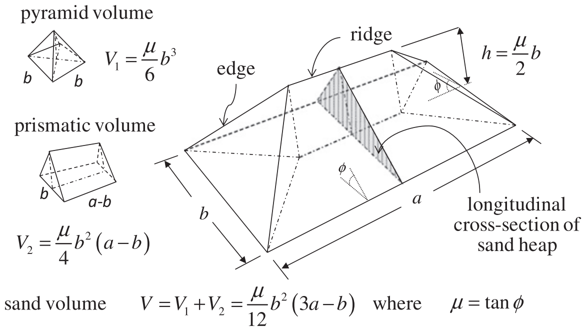


**Fig. 2.** Volume of sand heap deposited on a rectangular stiff base bounded by four stable slope surfaces and inclined horizontally along the angle of repose.