Journal of the Mechanics and Physics of Solids 57 (2009) 1718–1732

Contents lists available at ScienceDirect

Journal of the Mechanics and Physics of Solids

journal homepage: www.elsevier.com/locate/jmps

Shape and topology sensitivity analysis for cracks in elastic bodies on boundaries of rigid inclusions

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ARTICLE INFO

Article history:
Received 22 October 2008
Received in revised form
28 June 2009
Accepted 5 July 2009

Keywords:
Rigid inclusion
Crack growth
Contact condition
Shape sensitivity analysis
Topological derivatives

ABSTRACT

We consider an elastic body with a rigid inclusion and a crack located at the boundary of the inclusion. It is assumed that nonpenetration conditions are imposed at the crack faces which do not allow the opposite crack faces to penetrate each other. We analyze the variational formulation of the problem and provide shape and topology sensitivity analysis of the solution in two and three spatial dimensions. The differentiability of the energy with respect to the crack length, for the crack located at the boundary of rigid inclusion, is established.

1. Introduction

The problem associated with cracks in elastic bodies on boundaries of rigid inclusions appears in a vast number of applications in civil, mechanical, aerospace, biomedical and nuclear industries. In particular, some classes of materials are composed by a bulk phase with inclusions inside. When the inclusions are much stiffer than the bulk material, we can treat them as rigid inclusions. In addition, it is quite common to have cracks between both phases. Thus, in this paper we deal with the mechanical modeling as well as the shape and topology sensitivity analysis associated with the limit case of rigid inclusions in elastic bodies with a crack at the interface.

The mechanical modeling is based on the assumption of nonpenetration conditions at the crack faces between the elastic material and the rigid inclusion, which do not allow the opposite crack faces to penetrate each other, leading to a new class of variational inequalities. For the sensitivity analysis, we attempt to find the shape derivative of the elastic energy with respect to the perturbations of the crack tip. We also obtain the topological derivatives of the energy shape functional associated with the nucleation of a smooth imperfection in the bulk elastic material. These quantities are very important in design procedures and in numerical solution of some inverse problems. Both the analysis and the shape and topology optimization of this class of problems seem to be new and very useful from the mathematical and also the mechanical points of view.

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0022-5096/$ - see front matter
doi:10.1016/j.jmps.2009.07.003

The paper is organized as follows. The problem formulation associated with cracks in elastic bodies on boundaries of rigid inclusions is presented in Section 2. Some results concerning shape sensitivity analysis with respect to the perturbations of the crack tip are given with all details in Section 3. The topological derivatives associated with the energy shape functional are calculated in Section 4. We provide some closed formulas for the case of nucleation of spherical holes in 3D and circular elastic inclusions in 2D. In this last case, we present the limit cases in which the elastic inclusion becomes a hole (void) and also a rigid inclusion.

2. Problem formulation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ, and $\omega \subset \Omega$ be a subdomain with smooth boundary Ξ such that $\overline{\omega} \cap \Gamma = \emptyset$. We assume that Ξ consists of two parts γ and $\Xi \backslash \gamma$, $meas(\Xi \backslash \gamma) > 0$, where γ is a smooth 2D surface described as

$$x_i = x_i(y_1, y_2) \quad (y_1, y_2) \in D, \ \ i = 1, 2, 3$$

with bounded domain $D \subset \mathbb{R}^2$ having a smooth boundary ∂D, and the rank of the matrix $\partial x/\partial y$ is equal to 2.

Denote by $\nu = (\nu_1, \nu_2, \nu_3)$ a unit outward normal vector to Ξ, see Fig. 1. The subdomain ω is assumed to correspond to a rigid inclusion, and the surface γ describes a crack located on Ξ. Domain $\Omega \backslash \overline{\omega}$ corresponds to the elastic part of the body. For the further use we introduce the space of infinitesimal rigid displacements

$$R(\omega) = \{\rho = (\rho_1, \rho_2, \rho_3) \mid \rho(x) = Bx + C, \ x \in \omega\},$$

where

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{pmatrix}, \quad C = (c^1, c^2, c^3); \quad b_{ij}, c^i = const, \quad i,j = 1, 2, 3.$$

Denote $\Omega_\gamma = \Omega \backslash \overline{\gamma}$. Problem formulation describing an equilibrium of the elastic body with the rigid inclusion ω and the crack γ is as follows. In the domain Ω_γ, we have to find functions $u = (u_1, u_2, u_3)$, $u = \rho_0$ on ω; $\rho_0 \in R(\omega)$; and in the domain $\Omega \backslash \overline{\omega}$ we have to find functions $\sigma = \{\sigma_{ij}\}$, $i,j = 1, 2, 3$, such that

$$-\operatorname{div} \sigma = F \quad \text{in } \Omega \backslash \overline{\omega}, \tag{1}$$
$$\sigma - A\varepsilon(u) = 0 \quad \text{in } \Omega \backslash \overline{\omega}, \tag{2}$$
$$u = 0 \quad \text{on } \Gamma, \tag{3}$$
$$(u - \rho_0) \cdot \nu \geq 0 \quad \text{on } \gamma^+, \tag{4}$$
$$\sigma_\tau = 0, \quad \sigma_\nu \leq 0 \quad \text{on } \gamma^+, \tag{5}$$
$$\sigma_\nu (u - \rho_0) \cdot \nu = 0 \quad \text{on } \gamma^+, \tag{6}$$

$$-\int_\Xi \sigma\nu \cdot \rho = \int_\omega F \cdot \rho \quad \forall \rho \in R(\omega). \tag{7}$$

Here $F = (F_1, F_2, F_3) \in L^2(\Omega)$ is a given function,

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu,$$
$$\sigma_\tau = (\sigma_\tau^1, \sigma_\tau^2, \sigma_\tau^3), \quad \sigma\nu = \{\sigma_{ij}\nu_j\}_{i=1}^{i=3},$$
$$\varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1, 2, 3.$$

All functions with two lower indices are assumed to be symmetric in those indices. Summation convention over repeated indices is accepted throughout the paper. Elasticity tensor $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2, 3$, is given, and it satisfies the usual symmetry and positive definiteness properties,

$$a_{ijkl} = a_{klij} = a_{jikl}, \quad a_{ijkl} \in L^\infty(\Omega), \quad i, j, k, l = 1, 2, 3,$$

$$a_{ijkl}\xi_{kl}\xi_{ij} \geq c_0|\xi|^2 \quad \forall \xi_{ij} = \xi_{ji}, \quad c_0 = const.$$

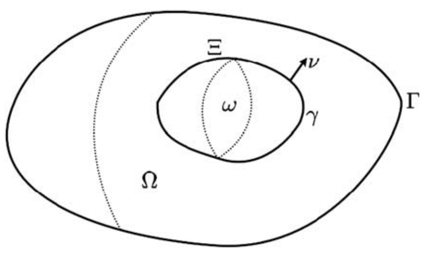

Fig. 1. Domain Ω with rigid inclusion ω.