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A general hyperelastic model for incompressible fiber-reinforced elastomers

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ABSTRACT

This work presents a new constitutive model for the effective response of fiber-reinforced elastomers at finite strains. The matrix and fiber phases are assumed to be incompressible, isotropic, hyperelastic solids. Furthermore, the fibers are taken to be perfectly aligned and distributed randomly and isotropically in the transverse plane, leading to overall transversely isotropic behavior for the composite. The model is derived by means of the "second-order" homogenization theory, which makes use of suitably designed variational principles utilizing the idea of a "linear comparison composite." Compared to other constitutive models that have been proposed thus far for this class of materials, the present model has the distinguishing feature that it allows consideration of behaviors for the constituent phases that are more general than Neo-Hookean, while still being able to account directly for the shape, orientation, and distribution of the fibers. In addition, the proposed model has the merit that it recovers a known exact solution for the special case of incompressible Neo-Hookean phases, as well as some other known exact solutions for more general constituents under special loading conditions.

1. Introduction

Fiber-reinforced, elastomer-matrix composites constitute a broadly utilized class of materials in engineering applications. In addition, fiber-reinforced-type morphologies appear naturally in a number of other "soft" matter systems of current interest. Prominent examples include nano-structured thermoplastic elastomers (see, e.g., Honeker and Thomas, 1996; Honeker et al., 2000) and soft biological tissues (see, e.g., Finlay et al., 1998; Quapp and Weiss, 1998). It is often the case that such fiber-reinforced "soft" materials are subjected to finite deformations, and it is therefore of practical interest to develop constitutive models for their mechanical behavior under such loading conditions. Beyond accounting for finite deformations, it is also desirable that these models incorporate full dependence on the constitutive behavior of the constituents (i.e., the matrix phase and the fibers), as well as on their spatial arrangement (i.e., the microstructure). In this paper, we will consider fiber-reinforced elastomers with *hyperelastic* matrix and fiber phases. In addition, we will restrict attention to microgeometries with a single family of aligned fibers which are taken to be initially circular in cross section and *randomly* and *isotropically* distributed in the undeformed configuration.

A variety of efforts have been pursued over the past few decades to model the effective behavior of fiber-reinforced hyperelastic materials. In terms of phenomenological approaches, there is the pioneering theory of Spencer (1972), in the

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context of which the fibers are treated as inextensible material line elements. Other more sophisticated phenomenological models have been constructed by augmenting existing isotropic stored-energy functions with additional terms depending on the transversely isotropic invariants associated with the fiber direction (Spencer, 1984). Examples include the models proposed by Qiu and Pence (1997), Merodio and Ogden (2005), Horgan and Saccomandi (2005), and Gasser et al. (2006). Although these models possess a number of desirable features, and in particular, they are simple and can be "calibrated" to become macroscopically unstable—via loss of strong ellipticity—for loading conditions where such instabilities are expected to occur from physical experience (Triantafyllidis and Abeyaratne, 1983), their predictive capabilities for the general response of actual fiber-reinforced elastomers remain limited. In a separate effort—essentially making use of a micromechanics approach—Guo et al. (2006) have proposed a hyperelastic model for fiber-reinforced elastomers with *incompressible Neo-Hookean* matrix phases.

On the other hand, homogenization approaches have also been used to obtain bounds and estimates for the constitutive response of these materials. In particular, there is the simple, microstructure-independent, Voigt-type bound (Ogden, 1978), and the polyconvex Reuss-type lower bound (Ponte Castañeda, 1989), as well as an estimate put forward by deBotton (2005) and deBotton et al. (2006) for fiber-reinforced elastomers with *incompressible Neo-Hookean* phases. One of the strengths of the later model (deBotton et al., 2006) is that it predicts the exact effective response of composites with the "composite cylinder assemblage" microstructure, when subjected to axisymmetric and antiplane shear loadings. Based on the "second-order" homogenization procedure (Ponte Castañeda, 1996; Ponte Castañeda and Tiberio, 2000), Lahellec et al. (2004) proposed a constitutive model, for the transverse response of incompressible hyperelastic fiber-reinforced elastomers with *periodic* microstructures, and made successful comparisons with experimental and numerical results. In addition, by making use of the more recent "second-order" homogenization theory (Ponte Castañeda, 2002; Lopez-Pamies and Ponte Castañeda, 2006a), Lopez-Pamies and Ponte Castañeda (2006b) obtained closed-form estimates for the transverse in-plane response of incompressible elastomers reinforced with *randomly* distributed, *rigid* fibers, while Brun et al. (2007) provided more general estimates for fiber-reinforced elastomers with *compressible*, isotropic (matrix and fiber) phases and *periodic* microstructures.

In this paper, we will make use of the second-order homogenization theories (Ponte Castañeda and Tiberio, 2000; Lopez-Pamies and Ponte Castañeda, 2006a) to construct a complete three-dimensional constitutive model for the overall behavior of fiber-reinforced elastomers with *incompressible*, isotropic matrix and fiber phases and *random* microstructures. More specifically, the constitutive behaviors of the matrix and fibers are assumed to be characterized by *generalized Neo-Hookean* models. This class of materials is sufficiently general to model many types of real elastomers (see, e.g., Gent, 1996; Boyce and Arruda, 2000) and, at the same time, is sufficiently simple to lead to analytical results. Furthermore, the fibers are assumed to be perfectly aligned and to be distributed randomly and isotropically in the transverse plane, leading to overall transversely isotropic behavior for the composite. The main result of this paper is given by expression (38), together with expressions (39)–(42), which provide estimates for the effective stored-energy function of the composite materials of interest.

It is relevant to mention that the two "second-order" homogenization methods (Ponte Castañeda, 1996, 2002) were established on the common basis that available estimates for the effective behavior of (suitably constructed) linear composites can be converted into corresponding estimates for the effective behavior of non-linear composites. They both have the capability to account for statistical information on the initial microstructure beyond the volume fraction, as well as for its evolution, resulting from the applied finite deformations. This point is crucial as the evolution of the microstructure may have a significant geometric softening or stiffening effect on the overall response of the material, which, in turn, may lead to the possible development of macroscopic instabilities. The first method, when it works, is simpler to use than the second. When the first method fails, the second method, using additional information about the field fluctuations, can deliver improved results at the expense of a somewhat heavier implementation. Finally, it is important to mention that in addition to the already-mentioned applications to fiber-reinforced elastomers, these homogenization methods can be employed more generally, and have already been used, for example, to construct constitutive models for the overall response of porous elastomers (Lopez-Pamies and Ponte Castañeda, 2004; Michel et al., 2007).

2. Problem formulation

Consider a specimen occupying a volume Ω_0 with boundary $\partial\Omega_0$ in the reference (undeformed) configuration, and made up of a single family of aligned, cylindrical fibers with circular cross section, distributed randomly and isotropically (in the transverse plane) in a matrix phase. The orientation of the fibers in the reference configuration is taken to be characterized by the unit vector **N**. Furthermore, it is assumed that the average diameter of the fibers is much smaller than the size of the specimen and the scale of variation of the applied load.

Both the matrix (phase 1) and the fibers (phase 2) are assumed to be made up of (different) homogeneous hyperelastic materials. Their constitutive behaviors are characterized by stored-energy functions $W^{(1)}$ and $W^{(2)}$, respectively, which are assumed to be *objective*, *isotropic*, strictly rank-one convex (strongly elliptic) functions of the deformation gradient tensor **F**. Here, we will restrict our attention to stored-energy functions $W^{(r)}$ for the phases ($r = 1, 2$) of the form

$$W^{(r)}(\mathbf{F}) = g^{(r)}(I) + h^{(r)}(J) + \frac{\kappa^{(r)}}{2}(J-1)^2, \tag{1}$$