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Minimum principles for the trajectories of systems governed by rate problems

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Abstract

Recently, Mielke and Ortiz [2007. A class of minimum principles for characterizing the trajectories of dissipative systems, ESAIM Control Optim. Calc. Var., in press] have proposed a variational reformulation of evolutionary problems that characterizes *entire trajectories* of a system as minimizers of certain *energy–dissipation* functionals. In this paper we present two examples of energy–dissipation functionals for which relaxations and optimal scalings can be rigorously derived. The first example concerns a one-dimensional bar characterized by a quadratic dissipation function and a bistable energy density; the second example concerns the coarsening kinetics of island growth in thin films exhibiting a preferred slope. In both cases, we present closed-form relaxations in the local limit of the problem and optimal scaling relations for the nonlocal problems. The relaxations rigorously characterize macroscopic properties of complex microstructural evolution by means of well-posed effective problems. The scaling relations rigorously characterize some average properties of the coarsening kinetics of the systems and lead to predictions on the growth exponents.

Keywords: Minimum principles; Rate problems; Dissipative systems; Metastability; Relaxation

1. Introduction

The evolution of many physical systems is the result of a balance between dissipative and energetic forces. This balance results in *evolutionary equations* that govern the time dependence of the state of the system. For linear systems, the resulting equations are parabolic. For these problems there is a vast literature that establishes fundamental properties of the solutions, including existence and uniqueness. Furthermore, the problem of determining the *rate* of change of the system given its state often has a variational structure that is well known in specific areas of application.

However, this classical treatment breaks down when applied to strongly nonlinear systems that develop evolving microstructures. In these cases, the energy of the system lacks lower semicontinuity and equilibrium solutions that minimize the energy do not exist in general. It is often possible, however, to characterize the

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stable equilibria of the state by means of minimizing sequences whose energy is arbitrarily small. In order to satisfy displacement boundary conditions, these sequences oscillate on increasingly finer scales but possess stable statistics or Young measures (cf. e.g., Müller, 1999). In a dynamical setting, nonexistence is often replaced by the existence of infinitely many solutions, as shown by Höllig (1983) and later refined by Zhang (2006a, b) using methods developed in Müller and Šverák (1996, 1999) and Kirchheim (2002). In addition, the state of the system is defined in terms of Young measures only, which renders the rate problem ill-defined, and the classical approach breaks down altogether.

One possible—and by now standard—way to circumvent this difficulty is by recourse to time-discretization (cf. e.g., Ortiz and Repetto, 1999; Ortiz et al., 2000; Miehe et al., 2002; Carstensen et al., 2002; Aubry and Ortiz, 2003; Mielke, 2004). This approach is effective when a persistent microstructure, i.e., one having a Young measure constant in time, is built into the system. Examples include ductile single crystals deforming under monotonic or cyclic proportional loading. In these cases the deformation of the crystal is governed by deformation theory of plasticity and conventional tools of the calculus of variations, such as relaxation and Γ-convergence (cf. e.g., DalMaso (1993)), apply.

We recall that, in problems that give rise to microstructure but lack an intrinsic length scale, relaxation results in the definition of an effective or macroscopic problem. The effective problem is well-posed and accounts for microstructure in an effective manner and without loss of information. In particular, given a solution of the effective problem it is always possible to construct minimizing sequences of the unrelaxed functional whose local averages match the macroscopic solution. In problems that exhibit microstructure and possess an intrinsic length scale, it is sometimes possible to derive *optimal scaling* relations that characterize the response of the system as a function of the parameters of the problem; see, e.g., Kohn and Müller (1992, 1994), Choksi et al. (1999) and Conti and Ortiz (2005).

However, time-incremental formulations suffer from the same *restart* difficulty as rate problems, namely, that the incremental problem becomes ill-defined when a microstructure has been previously established in the system and the initial conditions for the time step are only defined in the sense of Young measures.

Recently, Mielke and Ortiz (2007) have proposed a variational reformulation of evolutionary problems that characterizes *entire trajectories* of the system as minimizers of certain *energy–dissipation* functionals, thus entirely circumventing the *restart* problem of classical rate and time-incremental problems. The resulting formulation is based on the time integration of scaled energy and dissipation functionals with an exponentially decaying weight, and may be regarded as an elliptic-in-time regularization (a similar elliptic regularization was proposed by Ilmanen, 1994, for the specific case of mean-curvature flow). Because of the minimizing property of the trajectories of the system, tools of the calculus of variations such as the direct method for establishing existence, relaxation and optimal scaling can again be applied to the problem. When applied to energy–dissipation functionals, relaxation results in the definition of an effective or macroscopic evolutionary problem. The effective problem is well-posed and accounts for microstructure evolution in an effective manner and without loss of information. In particular, given a macroscopic trajectory of the system it is always possible to construct minimizing sequences of trajectories of the unrelaxed energy–dissipation functional whose local space–time averages match the macroscopic trajectory. In problems that exhibit microstructure and possess intrinsic length and time scales, *optimal scaling* relations, when known, characterize the effective coarsening kinetics of the system, including growth exponents.

To date, this program has only been applied to energy–dissipation functionals of the rate-independent type, for which general abstract results regarding existence and relaxation have been derived (Mielke and Ortiz, 2007). The purpose of this paper is to present two rate-dependent examples for which relaxations and optimal scalings can be rigorously derived. The first example concerns a one-dimensional bar characterized by a quadratic dissipation function and a bistable energy density, and is modeled after the Kohn–Müller model of branching in martensite (Kohn and Müller, 1992, 1994). The second example concerns the coarsening kinetics of island growth in thin films exhibiting a preferred slope, and is modeled after the Ortiz–Repetto–Si (ORS) model of kinetic roughening and coarsening in thin films Ortiz et al. (1999). In both cases, the local limit of the problem is amenable to closed-form relaxation, Sections 3.1 and 4.1; and optimal scaling relations can be rigorously derived for the nonlocal problems, Sections 3.2 and 4.2. These examples illustrate how the energy–dissipation formalism can indeed shed light into microstructural evolution and facilitate the definition of effective problems governing the macroscopic evolution of the system.