Mechanics of Materials 95 (2016) 192–203

Contents lists available at ScienceDirect

Mechanics of Materials

journal homepage: www.elsevier.com/locate/mechmat

Analytic solutions for the stress field in static sandpiles

M. Angelillo[a,*], E. Babilio[b], A. Fortunato[a], M. Lippiello[b], A. Montanino[c]

[a] Department of Civil Engineering, University of Salerno, Via Ponte Don Melillo, Fisciano 84084, SA, Italy
[b] Department of Structures for Engineering and Architecture, University of Naples "Federico II", Via Forno Vecchio 36, Napoli 80134, Italy
[c] Centro di Simulazione Numerica Avanzata (CeSNA), Istituto Universitario di Studi Superiori (IUSS), Pavia, Italy

ARTICLE INFO

Article history:
Received 26 June 2015
Revised 28 January 2016
Available online 4 February 2016

Keywords:
Granular matter
Coulomb friction
Analytical equilibrated stress fields
Arching

ABSTRACT

In the present paper we propose a new class of analytical solutions for the equilibrium problem of a prismatic sand pile under gravity, capturing the effects of the history of the sand pile formation on the stress distribution. The material is modeled as a continuum composed by a cohesionless granular material ruled by Coulomb friction, that is a material governed by the Mohr–Coulomb yield condition. The closure of the balance equations is obtained by considering a special restriction on stress, namely a special form of the stress tensor relative to a special curvilinear, locally non-orthogonal, reference system.

This assumption generates a class of closed-form equilibrium solutions, depending on three parameters. By tuning the value of the parameters a family of equilibrium solutions is obtained, reproducing closely some published experimental data, and corresponding to different construction histories, namely, for example, the deposition from a line source and by uniform raining. The repertoire of equilibrated stress fields that we obtain in two special cases contains an approximation of the *Incipient Failure Everywhere* (IFE) solution and a closed-form description of the arching phenomenon.

1. Introduction

Granular piles in static equilibrium have attracted much interest due to their rich physical phenomenology and the number of curious effects that they exhibit. The most popular among these effects is the appearance of a stress dip at the center of the pile, that was detected experimentally and described and interpreted by several papers in the late 1990s . The main result of these studies (among which we recall the comprehensive work of Wittmer et al. (1996) and the extensive references cited therein) is that the dip is a result of arching. Describing the degree of arching and predicting its dependence on the construction history was, at that time, still an open issue.

Such a challenge was taken up lately by a number of researchers, among which we recall Michalowski and Park (2004a, 2004b); 2005), Pipatpongsa and Siriteerakul (2010), Pipatpongsa et al. (2010), which propose some analytic stress solutions, and Bierwisch et al. (2009), Sibille et al. (2015) and Zhu et al. (2008), which adopt numerical approaches based on discrete element simulations.

In the present paper we propose a new analytical solution to the equilibrium problem of a prismatic sand pile under gravity. Our main objective is to give closed-form solutions of the equilibrium equations, capable of describing the different degrees of arching that are determined by different construction histories. It is a fact, for example as shown by Vanel et al. (1999), that the pressure dip beneath the pile can have different profiles or even almost disappear depending on the loading history. Such an effect is

* Corresponding author. Tel.: +39 089964073; fax: +39 089964045.
E-mail addresses: mangelillo@unisa.it (M. Angelillo), enrico.babilio@unina.it (E. Babilio), a.fortunato@unisa.it (A. Fortunato), maria.lippiello@unina.it (M. Lippiello), andrea.montanino@iusspavia.it (A. Montanino).

http://dx.doi.org/10.1016/j.mechmat.2016.01.015
0167-6636/

captured by our closed-form solution just by tuning three parameters.

The material is modeled as a continuum composed by a cohesionless granular material. The modeling strategy is based on some simplifying assumptions:

- There is a stress tensor **T** which is well defined as a local average over many grains.
- The particle are hard (that is are rigid), therefore no elastic strain variable exists and the static frictional forces are indeterminate.
- Friction is ruled by Coulomb linear law, that is the stress is restricted by inequalities depending on a single coefficient (see the inequalities (2) below).
- The construction history determines the grain arrangement and packing, that is the local restrictions ruling the stress.

In the model there is no trace of elasticity or of any constitutive behaviour or dependence of the stress on the strain or on the strain rate. The main tool that we use to select a particular equilibrium solution is represented by the choice of a special curvilinear system (θ_1, θ_2), with the lines θ_2 directed vertically and the lines θ_1 directed along curved trajectories. This family of curved lines can be chosen, in principle, in an infinite number of different ways, but we explore just the easiest possibilities namely: hyperbolas and parabolas. The closure condition that we adopt to make the problem statically determined, consists in assuming that the tangential stress component, in this special reference system, is zero, that is the vertical load (the body load) is taken by normal stresses directed as the vertical lines θ_2 and as the curved lines θ_1. The ratio between the load that is taken by the columns, and the load that is taken by the arches can be regulated by tuning three parameters.

In particular, in the case in which the curved lines are parabolas, we find two special solutions that, in terms of base pressures, resemble closely the experiments by Vanel et al. (1999).

2. The Coulomb friction model

As a first approximation to the behaviour of dry granular masses, a cohesionless material ruled by Coulomb friction, that is a material governed by the Mohr–Coulomb yield condition, can be adopted. This crude unilateral model material that idealizes the real material as indefinitely strong if compressed within the yield cone, but incapable of sustaining stresses outside the yield surface, is perfectly rigid in compression, in the sense that no strain rates can occur if the stress is inside the limit surface. The material can exhibit unaccelerated flow when the stress belongs to the limit surface and accelerated motion must occur if the stress is outside the yield locus (see Lippiello, 2007).

If we restrict to statics, and exclude therefore the possibility of accelerated motion, the stress state is restricted to belong to the Mohr–Coulomb cone. In the plane case, denoting **T** the stress tensor, the Mohr–Coulomb restriction, in terms of the first and second stress invariants

$$\iota_1(\mathbf{T}) = \mathrm{tr}\,\mathbf{T} \quad \iota_2(\mathbf{T}) = \det \mathbf{T}, \tag{1}$$

can be written as follows:

$$\det \mathbf{T} - \cos^2(\varphi)\left(\frac{\mathrm{tr}\,\mathbf{T}}{2}\right)^2 \geq 0\,, \quad \mathrm{tr}\,\mathbf{T} \leq 0, \tag{2}$$

φ being the friction angle.

With Coulomb friction there is a linear relation between the normal component of stress and the maximum admissible tangential component that can be exerted on a given surface. When shear stress reaches this maximum value sliding on that surface becomes possible. Such a sliding is geometrically similar to the slipping occurring along discontinuity lines (concentrated shearing) in some perfectly plastic metallic materials, say plastic materials governed by the Mises yield condition. Energy is dissipated into heat during flow in both cases.

Unfortunately whilst the slipping flow rule of Mises materials is associative, the slipping of a Mohr–Coulomb material is not, since, for any value of the friction angle φ in the open interval $(0, \pi)$, such a strain rate is not orthogonal to the limit surface. Therefore the statics of frictional materials cannot be treated within the frame of Limit Analysis, unless the friction angle is $\pi/2$ (No-Tension materials, see Heyman, 1966; Angelillo, 1993; Fortunato, 2010; Angelillo et al., 2010, 2014a, 2014b, 2014c) or 0 (perfect unilateral fluids, see Chorin and Marsden, 2000; Dostal, 2009; Schechter and Bridson, 2012).

This means that, if, for a given structure under given loads, we find a stress field that is equilibrated with the loads and within the yield limits (that is a *statically admissible* stress field), we are not sure that the same structure, under the same loads, would not collapse for some special loading histories.

In the application we present here, we focus on the statical approach, namely, on assuming some closure restrictions on stress, we obtain statically admissible stress fields, that, in principle, can be attained by performing special loading histories. Actually, for granular masses the loading history is determined by the construction history, which determines the arrangement of *grains* and, therefore, the way in which the grains interact, that is, in the continuum model, the special material restrictions that are locally valid.

3. Formulation of the problem

The equilibrium of a prismatic sand pile of height h and base $2a$ (Fig. 1), and standing on a perfectly rigid rough plane, is considered. The angle of repose $\psi = \arctan(h/a)$ of the pile is usually assumed coincident with the friction angle of the material φ, though the case of a repose angle less than φ can be considered.

The typical construction history for a prismatic sand pile consists in pouring the material from a *line source*. Another typical construction history considers the pile constructed through uniform deposition (*uniform raining*).

3.1. Geometry and curvilinear coordinates

The relevant geometrical dimensions of a prismatic pile are its height h and its width $2a$. By reducing the problem to the plane case, the density ρ of the material is defined