Mechanics of Materials 69 (2014) 93–105

Contents lists available at ScienceDirect

Mechanics of Materials

journal homepage: www.elsevier.com/locate/mechmat

On some controversial issues in effective field approaches to the problem of the overall elastic properties

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ARTICLE INFO

Article history:
Received 28 December 2012
Received in revised form 15 July 2013
Available online 18 September 2013

Keywords:
Effective properties
Effective field
Multiphase composites

ABSTRACT

Several versions of the effective field method of finding the overall elastic properties of matrix composites are examined and compared. We focus on difficulties and uncertainties encountered by these methods, in particular in cases of anisotropic multiphase composites. It is demonstrated that the schemes are best formulated in terms of the compliance/stiffness contributions tensors: such formulation exposes roots of various inconsistencies, and clarifies relations between different versions of the method. Particular attention is paid to Maxwell's scheme which is shown to represent yet another version of the effective field method. The discussion can be extended to physical properties other than the elastic ones.

1. Introduction

We address some controversial issues related to homogenization of matrix composites, particularly the multiphase ones (containing mixtures of inhomogeneities of diverse shapes and properties). The simplest approach to the problem is the non-interaction approximation (NIA) whereby the inhomogeneities are treated as isolated ones placed in the externally applied field unperturbed by neighbors. At concentrations when interactions cannot be neglected, several approximate models have been proposed that can be classified as "one-particle" approximations: the interaction effect is simulated by placing inhomogeneities – treated as *isolated* ones – into certain "effective" environment. The latter can be chosen in one of two ways: either as *effective matrix* (possessing the yet unknown effective properties), or as *effective field* (stress or strain) that differs from the remotely applied one. For reviews, see, for example, Hashin (1983) and Markov (2000).

Various variants of the effective media approximation has been discussed in detail in the book of Milton (2002) and more recent papers of Benveniste and Milton (2010a, 2010b, 2011). In particular, they shown that the effective media scheme, being realizable, will always obey the Hashin–Shtrikman bounds, and that the effective field approaches may violate those bounds. We consider effective media approaches as well discussed in the literature and do not creating any controversial issues. The present paper focuses on the effective field models that are more physically grounded since they have direct interpretation in terms of stress superpositions whereby the effect of neighbors on a given inhomogeneity is expressed in terms of the field generated by the former at the site of the latter; the mentioned models make various simplifying assumptions on this field.

The simplest variant of the effective field method was proposed by Mori and Tanaka (1973); it became popular after Benveniste (1987) illustrated its use and clarified the underlying physical hypothesis. It takes the effective field acting on each inhomogeneity as the field average over the matrix. Then the effective properties can be obtained from the NIA results by replacing the remotely applied field by the matrix average one. The Kanaun–Levin's scheme (see Kanaun (1977), Levin (1976) and the

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0167-6636/$ - see front matter
http://dx.doi.org/10.1016/j.mechmat.2013.09.010

book of Kanaun and Levin (2008)) represents a refinement of Mori–Tanaka's scheme that is capable of accounting for finer details of microstructure, such as spatial statistics of centers of inhomogeneities.

Special place belongs to the Maxwell's (1873) model. It considers certain domain Ω containing a sufficient number of inhomogeneities (so that it can be treated as a representative volume element, RVE), and focuses on the far-field perturbation of the remotely applied field produced by this domain. This perturbation is calculated in two ways: (1) as a sum of far-field perturbations due to individual inhomogeneities treated as non-interacting ones, and (2) as generated by a homogenized domain Ω possessing the effective properties. Equating the results yields the effective constants.

Recently, Maxwell's scheme, originally developed for electrical conductivity of a matrix containing spherical inhomogeneities, has been extended to the elastic properties of a material containing either randomly oriented (McCartney and Kelly, 2008) or parallel (McCartney, 2010) ellipsoidal inhomogeneities of identical aspect ratios. Levin et al. (2012) applied it to the elastic, electric and poroelastic properties of a composite containing several families of ellipsoidal inhomogeneities, with aspect ratios identical within each family. We show, in the text to follow, that Maxwell's model actually represents a variant of the effective field method, as clarified by rewriting it in the form proposed by Sevostianov and Giraud (2013).

The effective field methods have their shortcomings. Predictions of the Maxwell's scheme depend on the shape of the homogenized domain Ω, and this dependence may be quite strong in anisotropic cases; at the same time, no clear guidance has been given on the choice of the shape. The Mori–Tanaka's scheme, being applied to multiphase composites, may violate the Hashin–Shtrikman bounds and, also, yield non-symmetric effective stiffnesses in certain cases of overall anisotropy (as well as other inconsistencies). To avoid the non-symmetry, Kanaun–Levin's scheme places inhomogeneities of different shapes or properties into different effective fields (this may not be easily reconciled with the principle of stress superpositions).

The present work clarifies these issues using the concept of property contribution tensors of inhomogeneities (see Kachanov and Sevostianov, 2005 for detail). The discussed issues are particularly relevant for mixtures of inhomogeneities of diverse shapes or properties.

2. Property contribution tensors and concentration parameters

The property contribution tensor of an inhomogeneity – that characterizes its contribution to the overall properties – is of the central interest here: summation over them gives the change in the effective properties due to the presence of inhomogeneities. They also clarify the issue of the proper concentration parameters in whose terms the effective property is to be expressed. This issue – and the very possibility to introduce concentrations parameters – is nontrivial: whereas in the simplest case of inhomogeneities of identical shapes the concentration parameter is the volume fraction, it is much less clear in practically important cases of *mixtures* of diverse shapes, as well as cases of orientation distributions more complex than fully random or perfectly parallel ones. In the context of the elastic properties, the property contribution of an inhomogeneity is given by its compliance- or stiffness contribution tensor.

The inhomogeneity contribution depends on the physical constants of the matrix and the inhomogeneity, and on the shape and orientation of the latter. It is also affected by interactions with neighbors; however, accounting for this factor amounts to solving the interaction problem; hence these tensors are defined in the framework of the NIA (that, strictly speaking, makes their use beyond the NIA not perfectly logical although this is routinely done in approximate schemes). Summing up these tensors over the inhomogeneities contained in a RVE yields the proper concentration parameter (that represents inhomogeneities according to their actual contributions to the physical property considered). Note that such parameters do not generally reduce to volume fractions.

The present work focuses on the linear elastic properties. We consider certain reference volume V containing an inhomogeneity. We represent, as usual, the strain per volume V as a sum

$$\boldsymbol{\varepsilon} = \boldsymbol{S}^0 : \boldsymbol{\sigma}^\infty + \Delta\boldsymbol{\varepsilon} \tag{2.1}$$

where $\boldsymbol{S}^0$ is the compliance tensor of the matrix and $\boldsymbol{\sigma}^\infty$ is the "remotely applied" stress (more precisely, the constant stress corresponding to homogeneous boundary conditions in tractions, $t_i = \sigma_{ij}^\infty x_j$). The extra strain, per V, due to an inhomogeneity of volume V_1 is a linear function of the applied stress:

$$\Delta\boldsymbol{\varepsilon} = \frac{V_1}{V}\boldsymbol{H} : \boldsymbol{\sigma}^\infty \tag{2.2}$$

where the fourth-rank tensor $\boldsymbol{H}$ is the compliance contribution tensor of the inhomogeneity per its unit volume. It depends on the elastic constants of the matrix and the inhomogeneity and the shape of the latter. It possesses the usual symmetries of the compliance tensor ($H_{ijkl} = H_{jikl} = H_{ijlk} = H_{klij}$). For multiple inhomogeneities – with interactions between them neglected – the extra strain due to their presence is a sum

$$\Delta\boldsymbol{\varepsilon} = \frac{1}{V}\sum V_k \boldsymbol{H}^{(k)} : \boldsymbol{\sigma}^\infty \tag{2.3}$$

so that the extra compliance, calculated in the NIA is given by

$$\Delta\boldsymbol{S} = \frac{1}{V}\sum V_k \boldsymbol{H}^{(k)} \tag{2.4}$$

This solves the problem in the NIA provided the $\boldsymbol{H}$-tensors of inhomogeneities – treated as isolated ones – are known. This explains the fundamental role of $\boldsymbol{H}$-tensors: for multiple inhomogeneities, it is *them* that have to be summed up.

The effective field models have the form of (2.3) with remotely applied field $\boldsymbol{\sigma}^\infty$ replaced by certain effective field $\boldsymbol{\sigma}^{eff}$. In other words, in the effective field models, the problem reduces to finding a fourth-rank tensor $\boldsymbol{M}$ that relates the two fields: