Mechanics Research Communications 69 (2015) 40–44

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

New solutions of classical problems in rigid body dynamics

H.M. Yehia*, E. Saleh, S.F. Megahid

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

ARTICLE INFO

Article history:
Received 20 September 2014
Accepted 23 May 2015
Available online 1 June 2015

Keywords:
Rigid body dynamics
Particular solutions
Motion of a rigid body in a liquid
Newtonian field
Attraction centre

ABSTRACT

The equations of motion of a rigid body acted upon by general conservative potential and gyroscopic forces were reduced by Yehia to a single second-order differential equation. The reduced equation was used successfully in the study of stability of certain simple motions of the body. In the present work we use the reduced equation to construct a new particular solution of the dynamics of a rigid body about a fixed point in the approximate field of a far Newtonian centre of attraction. Using a transformation to a rotating frame we also construct a new solution of the problem of motion of a multiconnected rigid body in an ideal incompressible fluid. It turns out that the solutions obtained generalize a known solution of the simplest problem of motion of a heavy rigid body about a fixed point due to Dokshevich.

1. Introduction

The equations of motion of a rigid body acted upon by general conservative potential and gyroscopic forces were reduced by Yehia to a single second-order differential equation [25,26]. The reduced equation was used successfully in the study of stability of certain simple motions of the body.

In the present work we use the reduced equation to construct a new particular solution of the dynamics of a rigid body about a fixed point in the approximate field of a far Newtonian centre of attraction. Using a transformation to a rotating frame we also construct a new solution of the problem of motion of a multiconnected rigid body in an ideal incompressible fluid. It turns out that the solutions obtained generalize a known solution of the simplest problem of motion of a heavy rigid body about a fixed point due to Dokshevich.

2. Classical problems of rigid body dynamics

The equations of motion of a rigid body about a fixed point can be written in the Euler-Poisson form [1]:

$$\begin{aligned} A\dot{p} + (C-B)qr &= \gamma_2 \frac{\partial V}{\partial \gamma_3} - \gamma_3 \frac{\partial V}{\partial \gamma_2} \\ B\dot{q} + (A-C)pr &= \gamma_3 \frac{\partial V}{\partial \gamma_1} - \gamma_1 \frac{\partial V}{\partial \gamma_3} \\ C\dot{r} + (B-A)pq &= \gamma_1 \frac{\partial V}{\partial \gamma_2} - \gamma_2 \frac{\partial V}{\partial \gamma_1} \end{aligned} \tag{1}$$

$$\dot{\gamma}_1 + q\gamma_3 - r\gamma_2 = 0,\ \dot{\gamma}_2 + r\gamma_1 - p\gamma_3 = 0,\ \dot{\gamma}_3 + p\gamma_2 - q\gamma_1 = 0 \tag{2}$$

where A, B, C are the principal moments of inertia, p, q, r are the components of angular velocity of the body and $\gamma_1, \gamma_2, \gamma_3$ are the components of the unit vector $\boldsymbol{\gamma}$ along the axis of symmetry of the force field, all being referred to the principal axes of inertia at the fixed point. The potential V depends only on the Poisson variables $\gamma_1, \gamma_2, \gamma_3$. Eqs. (1) and (2) admit three general first integrals:

$$\begin{aligned} I_1 &= \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + V = h, \ \text{the energy integral} \\ I_2 &= Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = f, \ \text{the areas integral} \\ I_3 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \ \text{the geometric integral} \end{aligned} \tag{3}$$

For Eqs. (1) and (2) with a given potential V to be integrable for arbitrary initial conditions, a fourth integral of motion must exist as a well-behaved function of the Euler–Poisson variables. As the problem is mostly non-integrable, the construction of any particular solution of the equations of motion acquires great importance.

* Corresponding author. Tel.: +20 1001780325; fax: +20 502246781.
E-mail addresses: hyehia@mans.edu.eg (H.M. Yehia), samifikri@mans.edu.eg (S.F. Megahid).

http://dx.doi.org/10.1016/j.mechrescom.2015.05.007
0093-6413/

2.1. The classical problem

The simplest version of the problem of motion of a rigid body about a fixed point described by Eqs. (1) and (2), usually termed "the classical problem", is characterized by the presence of a uniform gravity field. The potential in this version is

$$V = a\gamma_1 + b\gamma_2 + c\gamma_3 \tag{4}$$

where $a = Mgx_0$, $b = Mgy_0$, $c = Mgz_0$, M is the mass of the body, g is the gravity acceleration and (x_0, y_0, z_0) are the coordinates of the centre of mass referred to the system of principal axes at the fixed point.

This problem continued to be one of the most favourite problems for research for more than two centuries. It has become clear that the problem has only the following three general integrable cases, namely, Euler's [2], Lagrange's [3] and Kowalevski's [4] cases. Those are integrable for arbitrary initial conditions. There is also one conditional integrable case, named after Goriachev and Chaplygin, which is integrable only for motions on the zero level of the areas integral ($f = 0$).

Apart from general and conditional integral cases, there are 11 known particular solutions. Those are solutions of the Euler–Poisson equations valid under more restrictions on the initial conditions. A table of the 11 known cases of this problem was provided recently in [7].

2.2. The motion of a rigid body about a fixed point in a Newtonian field

The classical problem was considered in more than one generalized settings. The potential

$$V = a\gamma_1 + b\gamma_2 + c\gamma_3 + \frac{1}{2}n(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) \tag{5}$$

characterizes the motion about a fixed point of a rigid body subject to the Newtonian attraction of a far centre. In (5) a, b and c are the same as in (4) and $n = 3g/R$, where R is the distance between the centre of attraction and the fixed point (R is assumed very large, compared with the dimensions of the body).

The classical integrable cases of Euler and Lagrange were generalized to cases of the potential (5) [12,13], but Kowalevski and Goriachev-Chaplygin's cases proved incompatible with that generalization. The question of generalizing particular solutions mostly remains open.

2.3. The problem of motion of a rigid body in a liquid

The problem of motion by inertia of a rigid body bounded by a multi-connected surface in an infinite ideal incompressible fluid at rest at infinity generalizes the two problems described above. In its first version, the classical problem of the motion of a rigid body bounded by a simply-connected surface in an ideal incompressible fluid, it was explored in the works of Thompson and Tait and later developed by Kirchhoff, who formulated the equations of motion in the form known after his name and noted the simplest integrable case [10]. Clebsch [11] reformulated the equations in Hamiltonian form and obtained two integrable cases (for extended historical review, see, e.g. [8]). Steklov, Lyapunov and Chaplygin considered further Kirchhoff's equations and obtained other integrable cases.

Equations of motion of the full problem of the motion of a multi-connected (perforated) rigid body in a fluid circulating through perforations were formed in their final form in the Kirchhoff variables by Lamb [8].

In [9], a new form of the equations of the motion of a rigid body bounded by a multi-connected surface in an infinite ideal incompressible fluid is obtained by Yehia. Equations of the motion take the form

$$\dot{\omega}\mathrm{I} + \omega \times (\omega\mathrm{I} + \mathrm{k} + \gamma\bar{\mathrm{K}}) = \gamma \times (\mathrm{s} + \gamma\mathrm{J}),\ \dot{\gamma} + \omega \times \gamma = \mathbf{0} \tag{6}$$

where $\mathbf{I} = \mathrm{diag}(A, B, C)$, $\boldsymbol{\omega} = (p, q, r)$, $\bar{\mathrm{K}} = \frac{1}{2}(Tr\mathbf{K})\boldsymbol{\delta} - \mathbf{K}$, $\boldsymbol{\delta}$ being the unit matrix; $\mathbf{J}, \mathbf{K}$ are 3×3 constant symmetric matrices and $\mathbf{s}, \mathbf{k}$ are constant vectors. The system (6) is identical with the equation of the motion of a gyrostat (with gyrostatic moments $\mathbf{k}$) about a fixed point under the influence of a force field with potential $\mathrm{s}.\gamma + \frac{1}{2}\gamma\mathrm{J}.\gamma$ and gyroscopic forces whose moment is $-\omega \times \gamma\bar{\mathrm{K}}$.

The system (6) admits the three first integrals

$$\begin{aligned} I_1 &= \frac{1}{2}\omega.\omega\mathrm{I} + \mathrm{s}.\gamma + \frac{1}{2}\gamma\mathrm{J}.\gamma = h, \ \text{the energy integral} \\ I_2 &= \boldsymbol{\gamma}^2 = 1, \ \text{the analogue of geometric integral} \\ I_3 &= \left(\omega\mathrm{I} + \mathrm{k} + \frac{1}{2}\gamma\bar{\mathrm{K}}\right) \cdot \boldsymbol{\gamma} = f, \ \text{the cyclic integral} \end{aligned} \tag{7}$$

For the present problem, there are seven general and two conditional integrable cases. Very few particular solutions are known (see e.g. [22]).

3. Reduced equation in rigid body dynamics

The general problem of motion of a rigid body in a liquid described by equations of motion of the Euler–Poisson type (6), admit three general integrals of motion (7). In principle, this allows eliminating three of the six Euler–Poisson variables in virtue of those integrals and obtaining three first-order autonomous equations in the other three variables with respect to time as an independent variable. One more step can be taken, to eliminate time from the derivatives, and thus obtain two (non-autonomous) first order equations in two variables with respect to the third. Eventually, one can eliminate one of the variables to obtain a single second order equation in one variable with respect to the other.

Several authors tried to achieve maximal reduction of order to a single second-order equation using algebraic elimination processes.

Many trials were made by different authors to reduce the order of these equations using the known three classical integrals of the problem [14–17,19]. However, these trials either remained incomplete or they led to a situation which resembles that of the original system in depending on some unsolved constraints.

Kharlamov succeeded in reducing the problem of a gyrostat in the uniform field of gravity to a system of two differential equations of the first order [18]. His method depends on the elimination of the Poisson variables and cannot be generalized neither to more general potentials nor to cases when gyroscopic forces are present.

The ultimate solution of the problem of reduction of order of the equations of motion of a rigid body was achieved by Yehia in two works: for a gyrostat moving under action of arbitrary potential forces [25], and for a body acted upon by a general axi-symmetric combination of conservative potential and gyroscopic forces [26]. A single second-order differential equation in two of Poisson's variables γ_1, γ_3 is obtained for each case. This equation, connecting only geometric quantities, has proved very useful in certain qualitative and analytical studies of the motion of a rigid body, e.g. [23,24].

In the present work, we use the reduced equation of rigid body dynamics due to Yehia in building a new particular solution for the problem of motion of the body in the approximate field of a remote centre of Newtonian attraction. We will also use a transformation of the Euler–Poisson variables to generate a solution of the problem of motion of a rigid body in a liquid, which contains one parameter more than the last solution.