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Continuum-based stability of anisotropic and auxetic beams

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ABSTRACT

The critical buckling loads of pinned-pinned and cantilever beams are computed using the equations of three-dimensional elasticity rather than typical beam theories. These loads are influenced both by the nature of the assumed displacement field over the beam cross-section and by the inclusion of the terms from the full constitutive tensor. Of special interest are beams that are either anisotropic or auxetic. For anisotropic beams, an increased ratio of longitudinal to shear modulus for cantilevered beams increases the generation of shear buckling rather than flexural buckling. For isotropic auxetic beams, the values of Poisson ratio that define the limit between buckling loads that approach the classical buckling load from above or below are discussed.

1. Introduction

The buckling of beams under axial compression would make any short-list of well-studied problems of mechanics beginning with the achievement of Euler [1,2]. The voluminous works of Timoshenko [3] and Bažant and Cedolin [4] give extensive reviews of the historical developments along with the fundamental behavior of these shapes. More recent works have presented results using several beam theories [5,6].

Most theoretical representations of beam buckling represent the mechanics of deformation of the beam by reduced one-dimensional theories. These representations assume displacement fields that are relatively simple over the beam cross-section. There is of course very good reason for these simplified fields: they capture the basic behavior of the beams under these loading conditions. However, there are limits to these models as the beam transitions from being long and slender to geometries that are somewhat less so. The intent of this effort is to demonstrate at what level more complex fields might play a role.

The variance with the classical buckling loads is even more broad when the beam is composed of materials that are either anisotropic or auxetic. In the former case, the mismatch between the elastic moduli that usually dominate buckling behavior can become larger in magnitude. This increases the differences in predicted results between results of elementary beam theory, shear deformation theory, and elasticity theory. Auxetic materials, which are characterized by a negative Poisson ratio, can also generate behavior that deviates strongly from expected behavior as predicted by the elementary beam theory as it is usually applied to long, slender isotropic beams.

In this study, three-dimensional continuum-based models are used that generalize the deformation for a general constitutive tensor. Ritz-based approximations to virtual work statements are used to predict the buckling loads for the hinged-hinged beam and the cantilever beam under axial compression. The differences between the buckling load predictions for various theories are discussed for these cases.

2. One-dimensional solutions

For a beam under hinged-hinged (or simple-support) conditions with an applied axial load, the minimum load that initiates a buckled state is commonly referred to as the critical buckling load or the Euler buckling load, and is given by [3]

$$P_E = \frac{\pi^2 EI}{L^2} \tag{1}$$

This load is generated by solutions of equilibrium of the beam as represented by classical or Euler–Bernoulli beam theory, which assumes that only axial strains along the beam length are nonzero. This formula has excellent accuracy provided that the beam length is sufficiently large relative to the dimensions of the beam cross-section. In pure axial compression, the magnitude of the applied load is of course limited by the value that would cause axial

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deformation equal to the length of the beam, or an axial strain of −1. This implies that the load must be less than the cross-sectional area multiplied by the elastic modulus along the beam axis.

When the beam under consideration is somewhat more stocky, adjustments can be made in predicting the axial load that will generate failure. Humer [7] has recently given an excellent overview of the one-dimensional problem of an isotropic beam where shear deformation as modeled by Timoshenko beam theory is incorporated into the displacement field. The problem that includes extension, bending, and shearing is usually termed the generalized elastica. Humer denotes the three stiffnesses for these deformations via one-dimensional models that include the elastic longitudinal modulus E, the shear modulus G, the cross-sectional area A, and the second moment of the area about the axis of bending I. As the shear deformation formulation incorporates the Timoshenko assumption, this theory also requires the use of the shear coefficient k_s, which for the rectangular sections considered in this study is taken to be $k_s = 10(1+\nu)/(12+11\nu)$ where ν is the Poisson ratio. This formula was used by Humer [7] and was originally proposed by Cowper [8].

Humer generated a number of exact results for such a theory for several types of boundary condition. These were presented in terms of the dimensionless parameters given by

$$\lambda = L\sqrt{\frac{A}{I}} \tag{2}$$

that characterize the length, and by

$$\eta = \frac{k_s G}{E} = \frac{k_s}{2(1+\nu)} \tag{3}$$

that helps to characterize the relationship between the bending stiffness of the beam (dominated by E) and the shear stiffness (dominated by G). Humer also noted the possibility that the value of the Poisson ratio could be negative. Such solids, which came to be known as auxetic, have been discussed by Love [9] in 1944 and more recently by Lakes [10] for various foam structures and more recently by Lim [11,12] in buckling applications.

2.1. The hinged-hinged beam

For the case of both ends pinned, which is usually denoted either by the phrases simple support or hinged-hinged beam, Humer [7] found that the ratio between the various buckling loads and the classical Euler buckling load was given by

$$\frac{P_{cr}}{P_e} = \frac{1}{2}\frac{\eta}{\eta-1}\frac{\lambda^2}{\pi^2} \pm \sqrt{\frac{1}{4}\left(\frac{\eta}{\eta-1}\right)^2\frac{\lambda^4}{\pi^4} - n^2\frac{\eta}{\eta-1}\frac{\lambda^2}{\pi^2}} \tag{4}$$

Here n refers to the mode number associated with the buckling load, which is typically associated with the displaced shape given by

$$v(z) = A\sin\frac{n\pi z}{L} \tag{5}$$

Here $v(x)$ is the transverse displacement of the beam centroid along the length of the beam in the z direction.

As the beam length increases relative to the dimensions of the cross-section, nearly all beam theories trend to the value of the Euler buckling load. This means that the ratio of P_{cr}/P_E approaches the values of 1, 4, 9, and so on for modes 1, 2, and 3 since the buckling loads are a function of the square of the mode number [3]. In this work, most of the comparisons are confined to the first two buckling modes for the hinged-hinged beam.

2.2. The cantilever beam

For a beam under cantilever support, one end of the beam is fixed and the other is free excepting the uniform compression load. For this loading, Humer finds the buckling ratio for the shear deformation theory to be given by

$$\frac{P_{cr}}{P_e} = \frac{1}{2}\frac{\eta}{\eta-1}\frac{\lambda^2}{\pi^2} \pm \sqrt{\frac{1}{4}\left(\frac{\eta}{\eta-1}\right)^2\frac{\lambda^4}{\pi^4} - \frac{(2n-1)^2}{4}\frac{\eta}{\eta-1}\frac{\lambda^2}{\pi^2}} \tag{6}$$

In the thin-beam limit, the buckling load approaches the value given by the Euler theory in a manner similar to the hinged-hinged beam, but for this type of boundary condition combination the buckling loads are smaller. The first two values approach 1/4 and 9/4 of P_E for reasons explained by Humer [7].

For both support conditions, there is a shift in behavior of the Timoshenko beam model predictions as the parameter η tends to unity. Humer [7] explains this behavior in detail, but for purposes of this work it is sufficient to note that when $\eta = 1$ the critical buckling load changes in behavior for isotropic auxetic materials. As the beam length increases, P_{cr} approaches P_E either from below (when $\eta < 1$) or from above (when $\eta > 1$). This point is expanded upon in the sequel.

3. Continuum model: linear elasticity

A three-dimensional solid is considered whose cross-section coordinates are defined in the (x_1, x_2) or $(x-y)$ plane with a much larger dimension in the x_3 or z axis with a length of L. The beam is assumed to be composed of an anisotropic material whose principal material axes are aligned with the (x, y, z) axes. For purposes of this study, this material is assumed to be orthotropic.

3.1. Governing equations and weak form

The governing equations used for this study are the three-dimensional equations of linear elasticity with an orthotropic constitutive tensor. These are not solved explicitly at each point in the domain, but instead approximate solutions are sought for their weak form as expressed within the Principle of Virtual Work [13] or equivalent statements of total potential energy [4]. For so-called beam-columns under an axial force P, the total potential energy can be written as

$$\Pi = U - W \tag{7}$$

where U is the strain energy and W is the potential energy of the body force vector $\mathbf{f}$, the surface traction vector $\mathbf{t}$ excluding the axial force, and the applied axial force P, with

$$\begin{aligned} U &= \int_V \frac{1}{2}\sigma_{ij}\epsilon_{ij}dV \\ W &= \int_A \frac{P}{A}\Delta L dxdy + \int_V f_i u_i dV + \int_S t_i u_i dS \end{aligned} \tag{8}$$

Here σ_{ij} are used to denote components of Cauchy stress, ϵ_{ij} are the components of infinitesimal strain, u_i are the components of displacement, A is the cross-sectional area of the solid perpendicular to P, and ΔL is the distance over which the axial force P moves. In using indicial notation, it is assumed that the 1, 2, and 3 directions are ($x_1 = x$, $x_2 = y$, and $x_3 = z$) and that the long direction of the solid is the z-direction.

In an elasticity context, the axial force P is assumed to act through a uniform compressive normal traction P/A over the entire face of the beam cross-section. This means that each element of axial force $(P/A)dxdy$ moves an amount that varies with the