Mechanics Research Communications 69 (2015) 159–163

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

Analytical modelling of thermal stresses in anisotropic composites

Ladislav Ceniga *

Institute of Materials Research, Slovak Academy of Sciences, Watsonova 47, 040 01 Košice, Slovak Republic

ARTICLE INFO

Article history:
Received 16 December 2014
Received in revised form 29 May 2015
Accepted 2 June 2015
Available online 20 August 2015

Keywords:
Thermal stress
Two-component material
Anisotropic component
Analytical modelling

ABSTRACT

This paper represents a continuation of the author's previous work which deals with an analytical model of thermal stresses which originate during a cooling process of an anisotropic solid elastic continuum. This continuum consists of anisotropic spherical particles which are periodically distributed in an anisotropic infinite matrix. The infinite matrix is imaginarily divided into identical cubic cells with central particles. This multi-particle–matrix system represents a model system which is applicable to two-component materials of the precipitate–matrix type. The thermal stresses, which originate due to different thermal expansion coefficients of components of the model system, are determined within the cubic cell. The analytical modelling results from fundamental equations of continuum mechanics for solid elastic continuum (Cauchy's, compatibility and equilibrium equations, Hooke's law). This paper presents suitable mathematical procedures which are applied to the fundamental equations. These mathematical procedures lead to such final formulae for the thermal stresses which are relatively simple in comparison with the final formulae presented in the author's previous work which are extremely extensive. Using these new final formulae, the numerical determination of the thermal stresses in real two-component materials with anisotropic components is not time-consuming.

1. Introduction

This paper represents continuation of the author's paper [1] which deals with analytical modelling of thermal stresses in two-component materials of precipitate-matrix with anisotropic components. This analytical modelling results from fundamental equations of solid elastic continuum mechanics, i.e. Cauchy's, compatibility, equilibrium equations which are derived by the spherical coordinates $[r, \varphi, \nu]$, and from Hooke's law for an anisotropic solid continuum. Mathematical procedures presented in [1], which are applied to these fundamental equations, leads to dependences of tangential and shear stresses on a radial stress. Consequently, a final differential equation in terms of the radial stress is determined. Finally, a solution for the radial stress, and then solutions for the tangential and shear stresses are derived. However, these mathematical procedures lead to an extensive formula for the radial stress $\sigma_{rq} = C_{1q} r^{\lambda_{1q}} + C_{2q} r^{\lambda_{2q}}$ in the spherical particle ($q = p$) and the matrix ($q = m$). The exponents $\lambda_{1q}, \lambda_{2q}$ which represents real roots of a characteristic equation of the final differential equation are functions of 161 coefficients, $c_{1q}, \ldots, c_{161q}$, where c_{iq} ($i = 1, \ldots, 161$) is a function of the variables φ, ν [1]. Numerical determination of the radial, tangential and shear thermal stresses in anisotropic components of two-component materials is thus extremely time-consuming. The same is also valid for numerical determination of thermal-stress induced phenomena, e.g. cracking, limit state determination, and energy barrier, strengthening, which consider curve and surface integrals of thermal-stress induced energy density, respectively, as presented in [2,3].

This paper presents such mathematical procedures which lead to dependences of radial, tangential and shear stresses on a radial displacement. Consequently, a final differential equation in terms of the radial displacement is determined. Finally, a solution for the radial displacement, and then solutions for these stresses are derived. With regard to Eq. (27), the exponents $\lambda_{1q}, \lambda_{2q}$ are are functions of 51 coefficients, $c_{1q}, \ldots, c_{51q}$, where c_{iq} ($i = 1, \ldots, 51$) is a function of the variables φ, ν (see Eqs. (34), (35)).

This reduction of number of the coefficients leads to a significant reduction of time which is required for the numerical determination of these comprehensive analytical results (i.e. the thermal strains, thermal stresses and the thermal-stress induced phenomena mentioned above) for components of two-component materials [2,3]. Within the numerical determination, material parameters of a two-component material are substituted by their numerical values. Considering 15 formulae for the coefficients c_{1q},

* Tel.: +421 55 7922 432; fax: +421 55 7922 408.
E-mail addresses: lceniga@yahoo.com, ladislav.ceniga@yahoo.com, lceniga@imr.saske.sk
URL: http://www.imr.saske.sk.

http://dx.doi.org/10.1016/j.mechrescom.2015.06.017
0093-6413/

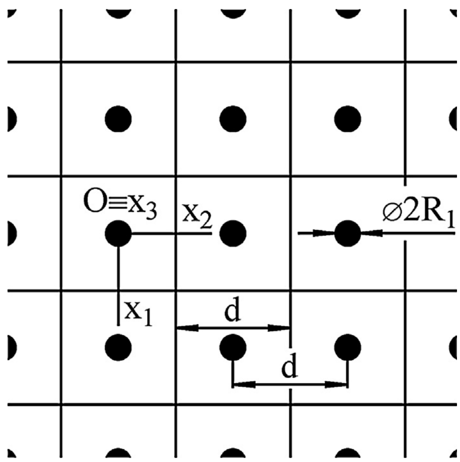

Fig. 1. The multi-particle–matrix system imaginarily divided into identical cubic cells with a central spherical particle in the point O of the Cartesian system $(Ox_1x_2x_3)$, where a dimension of the cubic cell is identical to the inter-particle distance d, and R_1 is the particle radius.

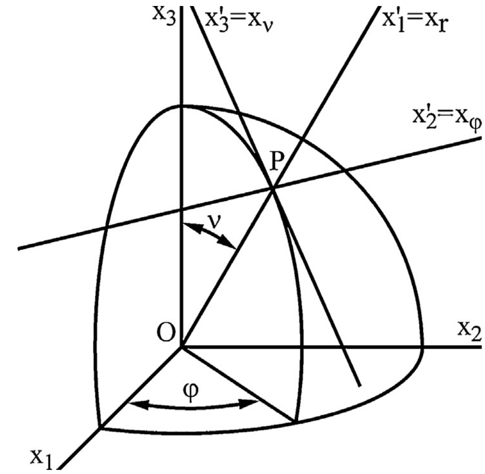

Fig. 2. The axes $x'_1 = OP$ and $x'_2 \| x_1x_2$, x'_3 defining radial and tangential directions regarding the Cartesian system $(Px'_1x'_2x'_3)$, respectively, and the arbitrary point P with a position determined by the spherical coordinates $[r, \varphi, \nu]$ regarding the Cartesian system $(Ox_1x_2x_3)$ (see Fig. 1). O is a centre of the spherical particle (see Fig. 1), and x'_2, x'_3 are tangents to a surface of a sphere with the radius $r = |\bar{OP}|$ representing length of the abscissa $|OP|$.

..., c_{51q} $(q = p, m)$ (see Eqs. (34), (35)), the substitution of numerical values for material parameters can be easily performed by several simple commands of a suitable programming language (e.g. Fortran, Pascal). Additionally, this substitution can be assumed to take shorter time in comparison with the use of numerical simulation methods (e.g. the finite element method). Precise numerical results of this analytical (mathematical) model are preferable to approximate numerical results of numerical simulation methods (e.g. the finite element method). Additionally, these precise numerical results (e.g. for stresses, strains) can thus represent boundary conditions for the numerical simulation methods which are used for complex models. Finally, this paper is also useful for those material scientists which are not experienced in numerical simulation methods.

Cell model. As presented in [1], the two-component materials of finite dimensions are replaced by the infinite multi-particle-matrix system shown in Fig. 1. This model system consists of anisotropic spherical particles which are periodically distributed in an anisotropic infinite matrix. The infinite matrix is imaginarily divided into identical cubic cells with a central spherical particle.

The cubic cell represents such part of the multi-particle–matrix system which is related to one spherical particle. The inter-particle distance d, the particle radius R_1 and the particle volume fraction $v = (4\pi/3)(R_1/d)^3 \in \langle 0, \pi/6\rangle$ [1], which represent parameters of the cubic cell, are microstructural parameters of two-component materials of the precipitate–matrix type. The thermal stresses are investigated within the cubic cell, and thus represent functions of the microstructural parameters. This 'cell approach' is usually used within mathematical procedures which are applied to analytical modelling of periodic model systems [4].

Coordinate system. The thermal stresses are derived in the arbitrary point P along the axes x'_1, x'_2, x'_3 of the Cartesian system $(Px'_1x'_2x'_3)$ (see Fig. 2). A position of the point P with respect to the Cartesian system $(Ox_1x_2x_3)$ is determined by the spherical coordinates $[r, \varphi, \nu]$, where $r = |\bar{OP}|$ is length of the abscissa $|OP|$, and O is a centre of the spherical particle (see Fig. 1). The axes x'_1 and x'_2, x'_3 represent radial and tangential directions, respectively.

The thermal stresses are then functions of $[r, \varphi, \nu]$. The spherical coordinates are used due to the fact that the particles of the multi-particle–matrix system are spherical. Due to the symmetry of the cubic cell, the thermal stresses are sufficient to be investigated within one eighth of the cubic cell, i.e. for $\varphi, \nu \in \langle 0, \pi/2\rangle$, where $r \in \langle 0, R_1\rangle$ for the spherical particle and $r \in \langle R_1, r_c\rangle$ for the cell matrix, respectively, and r_c is given by Eq. (33)).

Notation. The notation x'_1, x'_2, x'_3 instead of the conventional notation x_r, x_φ, x_ν (see Fig. 2), respectively, is used due to the notation which is used within the mathematical procedures in Section 3 (see e.g. the sum symbol $\sum_{i=1}^{3}$ in Eqs. (14)–(16)). Consequently, with regard to $x'_1 = x_r$, $x'_2 = x_\varphi$, $x'_3 = x_\nu$, the conventional notation for the thermal-stress induced radial displacement u_{rq} is replaced by u'_{1q}. Similarly, the radial stress σ_{rq} and the radial strain ε_{rq}, is replaced by σ'_{11q} and ε'_{11q}, respectively. The tangential stresses $\sigma_{\varphi q}$, $\sigma_{\nu q}$ and the tangential strains $\varepsilon_{\varphi q}$, $\varepsilon_{\nu q}$ are replaced by σ'_{22q}, σ'_{33q} and ε'_{22q}, ε'_{33q}, respectively. Finally, the shear stresses $\sigma_{r\varphi q}$, $\sigma_{r\nu q}$, $\sigma_{\varphi\nu q}$ and the shear strains $\varepsilon_{r\varphi q}$, $\varepsilon_{r\nu q}$, $\varepsilon_{\varphi\nu q}$ are replaced by σ'_{12q}, σ'_{13q}, σ'_{23q} and ε'_{12q}, ε'_{13q}, ε'_{23q}, respectively.

The radial displacements u'_{1p} for $q = p$ and u'_{1m} for $q = m$ are related to the spherical particle and cell matrix, respectively. The same is also valid for the radial stress σ'_{1q}, the tangential stresses σ'_{2q}, σ'_{3q} and the shear stresses σ'_{12q}, σ_{13q} in the spherical particle $(q = p)$ and cell matrix $(q = m)$.

2. Fundamental equations

The Cauchy's equations determine relationships between strains and displacements of an infinitesimal part of a solid continuum [5]. The equilibrium equations which are related to the axes x'_1, x'_2, x'_3 are based on a condition of the equilibrium of forces which act on sides of this infinitesimal part [5]. Due to the spherical coordinates $[r, \varphi, \nu]$, the infinitesimal part is represented by an infinitesimal spherical cap with the surfaces S_r and S_{r+dr} with the surface area $A_r = r^2\, d\varphi\, d\nu$ and $A_{r+dr} = (r + dr)^2 d\varphi\, d\nu$ at the radii r and $r + dr$, respectively. The axis x'_1 represents a normal of S_r and S_{r+dr}. The infinitesimal spherical cap in the arbitrary point P with the coordinates $[r, \varphi, \nu]$ exhibits the radial displacement u'_{1q} $(q = p, m)$ in the Cartesian system $(Px'_1x'_2x'_3)$. The analysis of the fact that the infinitesimal spherical cap exhibits (in the Cartesian system $(Px'_1x'_2x'_3)$) the radial displacement u'_{1q} only is presented in detail in