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Variational formulation of the static Levinson beam theory

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ABSTRACT

In this communication, we provide a consistent variational formulation for the static Levinson beam theory. First, the beam equations according to the vectorial formulation by Levinson are reviewed briefly. By applying the Clapeyron's theorem, it is found that the stresses on the lateral end surfaces of the beam are an integral part of the theory. The variational formulation is carried out by employing the principle of virtual displacements. As a novel contribution, the formulation includes the external virtual work done by the stresses on the end surfaces of the beam. This external virtual work contributes to the boundary conditions in such a way that artificial end effects do not appear in the theory. The obtained beam equations are the same as the vectorially derived Levinson equations. Finally, the exact Levinson beam finite element is developed.

1. Introduction

The beam and plate theories by Levinson are widely known in the literature [1,2]. Soon after their publication, Bickford gave a variational formulation for a beam theory [3], and Reddy for a plate theory [4], based on the displacement fields used by Levinson [1,2]. Due to the fact that their variational formulations led to different equilibrium equations than the vectorial derivations by Levinson, the beam and plate theories by Levinson have since then been considered quite often as "variationally inconsistent". In contrast to this common belief, we show in this study that the Levinson beam theory is actually variationally consistent with certain provisions. Hereafter, the scope is limited to static beam theories.

Little attention has been paid to the fact that the assumed displacement field, which is exactly the same for the Levinson and Reddy–Bickford beam theories, is exclusively an interior field. The use of interior kinematics means that the end effects that decay with distance from the ends of a beam are neglected by virtue of the Saint Venant's principle. Note that, for example, the Euler–Bernoulli and Timoshenko beam theories are interior beam theories. Furthermore, the well-known two-dimensional (2D) Airy stress function solutions for an end-loaded cantilever and a uniformly loaded simply-supported beam are interior solutions (see, e.g. [5]). The modeling of the end effects in such problems requires the use of the Papkovich–Fadle eigenfunctions [6]. In his vectorial formulation, Levinson accounted correctly for the interior nature of his beam theory by using only the classical interior load resultants – the bending moment and the shear force [1]. Consequently, the theory provides the exact interior elasticity solutions, for example, for the central axis deflection and the 2D stresses of an end-loaded cantilever. As will be shown, to properly account for the interior nature in an energy-based formulation of the Levinson beam theory, one has to grasp the idea that the interior stresses of the beam act as surface tractions on the lateral interior end surfaces of the beam. If the work due to these surface tractions is not taken into account, the obtained beam theory will exhibit artificial end effects.

The current study is organized as follows. In Section 2, the static Levinson beam theory and its consistency with the Clapeyron's theorem are considered. In Section 3, a consistent variational formulation for the Levinson beam theory is carried out. An exact Levinson beam finite element is developed in Section 4 and conclusions are presented in Section 5.

2. Levinson beam theory

2.1. Stress boundary conditions and displacement field

Fig. 1 presents a beam subjected to a uniform load q, which we have chosen as a representative loading case for our developments. The beam has a narrow rectangular cross-section of constant thickness t and the length and height of the beam are L and h, respectively. The load resultants M and Q stand for the bending moment and shear force, respectively. The positive directions for the coordinates, uniform load and the load resultants are according to Levinson [1]. In his derivation of the theory, Levinson assumed

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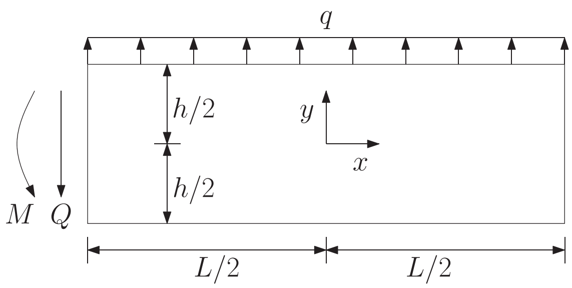

Fig. 1. Homogeneous isotropic beam with a narrow rectangular cross-section. The positive directions are according to [1]. The load resultants act at an arbitrary cross-section of the beam.

that (i) the transverse normal stress σ_y is zero throughout the beam and (ii) the Poisson effect (lateral contraction/expansion) is not accounted for. On the basis of these assumptions, and to satisfy the homogeneous boundary conditions $\tau_{xy}(x, \pm h/2) = 0$ on the upper and lower surfaces of the beam, Levinson obtained the 2D displacement field

$$U_x(x, y) = y\phi - \frac{4y^3}{3h^2}\left(\phi + \frac{\partial u_y}{\partial x}\right), \tag{1}$$

$$U_y(x, y) = u_y, \tag{2}$$

where $u_y(x)$ is the transverse deflection of the central axis of the beam and $\phi(x)$ is the clockwise positive rotation of the lateral cross-section at the central axis. The homogeneous boundary conditions are satisfied in a *strong* (pointwise) sense on the upper and lower surfaces of the beam. Levinson did not discuss in detail the stress boundary conditions on the lateral end surfaces of the beam, but it is crucial to note that in his theory the tractions at the beam ends are not specified at each point but only through the load resultants and, thus, the boundary conditions are imposed only in a *weak* sense [6]. The replacement of the true stress boundary conditions at the beam ends by the statically equivalent weak boundary conditions (load resultants) means that the exponentially decaying end effects of the beam are neglected by virtue of the Saint Venant's principle and only the interior solution of the beam is under consideration. The cross-sectional load resultants are calculated from the equations

$$M(x) = t\int_{-h/2}^{h/2} \sigma_x y\,dy, \quad Q(x) = t\int_{-h/2}^{h/2} \tau_{xy}\,dy \tag{3}$$

and can be used to impose natural interior boundary conditions at $x = \pm L/2$. The interior solution represents essentially a beam section which has been cut off from a complete beam far enough from the real lateral boundaries at which the true boundary conditions could be set.

2.2. General static solution

The static equilibrium equations for the Levinson beam theory are [1]

$$\frac{2}{3}GA\left(\phi + \frac{\partial u_y}{\partial x}\right) + \frac{EI}{5}\left(\frac{\partial^3 u_y}{\partial x^3} - 4\frac{\partial^2 \phi}{\partial x^2}\right) = 0, \tag{4}$$

$$\frac{2}{3}GA\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 u_y}{\partial x^2}\right) = -q. \tag{5}$$

where E and G are the Young's modulus and shear modulus, respectively. In addition, $A = ht$ and $I = th^3/12$ are the area of the cross-section and the second moment of the cross-sectional area, respectively. The kinematic and constitutive relations for the Levinson beam are

$$\epsilon_x = \frac{\partial U_x}{\partial x}, \quad \gamma_{xy} = \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x}, \tag{6}$$

$$\sigma_x = E\epsilon_x, \quad \tau_{xy} = G\gamma_{xy}, \tag{7}$$

respectively. The general polynomial solution to Eqs. (4) and (5) can be written as

$$u_y = c_1 + c_2 x + c_3 x^2 - c_4\frac{2x^3}{3h^2(1+\nu)} + \frac{qx^2}{120EI}[5x^2 - 12h^2(1+\nu)], \tag{8}$$

$$\phi = -c_2 - 2c_3 x + c_4\left[1 + \frac{2x^2}{h^2(1+\nu)}\right] - \frac{qx}{60EI}[10x^2 + 3h^2(1+\nu)]. \tag{9}$$

Note that the constant c_1 corresponds to rigid body translation in the y-direction and c_2 to a small counterclockwise rigid body rotation. We calculate the load resultants (3) using the stresses (7) and the general solution (8) and (9). Then, we can express the constants c_3 and c_4 in terms of the load resultants and substitute them back into the stresses (7) to obtain

$$\sigma_x = \frac{My}{I} + \frac{qy}{60I}(1+\nu)(20y^2 - 3h^2), \tag{10}$$

$$\tau_{xy} = \frac{Q}{8I}(h^2 - 4y^2). \tag{11}$$

Static bending solutions for the Levinson beam can be found in the papers by Levinson [1], Reddy et al. [7] and Reddy [8].

Finally, as the key item of this section, let us consider the strain energy of the beam and the external work done by the surface tractions. The strain energy and the external work due to the uniform load are

$$U = \frac{1}{2}\int_V (\sigma_x\epsilon_x + \tau_{xy}\gamma_{xy})dV, \quad W_q = \int_{-L/2}^{L/2} q u_y dx, \tag{12}$$

respectively. As a novel contribution, we consider the work due to the interior stresses on the lateral end surfaces of the interior beam and obtain

$$\begin{aligned} W_s = {} & t\int_{-h/2}^{h/2} \sigma_x(L/2, y)U_x(L/2, y)dy \\ & - t\int_{-h/2}^{h/2} \sigma_x(-L/2, y)U_x(-L/2, y)dy \\ & + t\int_{-h/2}^{h/2} \tau_{xy}(L/2, y)U_y(L/2, y)dy \\ & - t\int_{-h/2}^{h/2} \tau_{xy}(-L/2, y)U_y(-L/2, y)dy. \end{aligned} \tag{13}$$

By substituting the polynomial expressions for σ_x, ϵ_x, τ_{xy}, γ_{xy}, U_x and U_y according to the general solution (8) and (9) into Eqs. (12) and (13), we find that

$$2U - W_q - W_s = 0. \tag{14}$$

The above calculation shows that in static equilibrium the strain energy of the beam is equal to one-half of the work done by the surface tractions if they were to move (slowly) through their respective displacements. This is exactly in line with the *Clapeyron's theorem* (e.g. [9,10]). We conclude that the work done by the stresses on the