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Lower bound shakedown theorem for materials with internal rotation

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ABSTRACT

Materials with internal rotation belong to the class of polar materials, and they have a mechanical description suitable to model the behavior of microdevices, like MEMS, and materials such as grains, cellular solids and fiber-reinforced polymers. These types of materials have length scales that need a specific elastoplastic description. The shakedown theory is a useful tool for the analysis of mechanical components that are subjected to variable loads during some interval of time and for which only the limits of load variation are known. Although this may be a common mechanical situation, no shakedown theory is available for the analysis of materials with internal rotation. In this paper, a lower bound shakedown theorem for the analysis of materials with internal rotation is presented. The theorem is an extension of Melan's classical lower bound theorem and uses a linear elastic couple-stress theory combined with the first strain gradient plasticity of Fleck and Hutchinson. Considering a self-equilibrated couple-stress field, it is shown that the plastic dissipation is bounded and that the material shakes down for any given combinations of loads and moments. As a first attempt for a shakedown theory for this type of material, this theorem can give rise to variational formulations useful for numerical computations, similar to those previously used for the classical shakedown theory.

1. Introduction

The classical theories of elasticity and plasticity cannot fully describe the mechanical behavior of materials that are affected by strain gradients. For example, stress concentrations that are close to cavities, holes and cracks are not described adequately by the classical theory of elasticity. Similarly, several observed phenomena, such as shear band localization and the Hall–Petch effect, display a size effect not predicted by the conventional theory of plasticity. Microdevices, like microelectromechanical systems (MEMS), and materials such as grains, cellular solids and fiber-reinforced polymers also have important length scales that need a more accurate elastoplastic description to predict their mechanical properties.

The need for size-dependent mechanics endorsed a new branch of continuum mechanics, namely, the mechanics of generalized continua. New measures of strain, which are length related, are introduced in the continuum description, endowing the capture of the length scale effects. Couple-stress theory is a simplified alternative of micropolar theory [1]. In micropolar theory, every particle of a continuum is embedded in a microvolume, which can rotate independently from the surrounding medium. Thus, the motion of a particle is described by six degrees of freedom: three translational motions, as characterized in classical continuum mechanics, and three rotational motions, which describe the motion of the microvolume. The interactions between two adjacent surface elements are expressed by two vectors: a traction vector and a couple vector. In addition, the balance of angular momentum results in a non-symmetrical stress tensor [2,3].

The couple-stress theory is occasionally mentioned in the literature as indeterminate couple-stress theory [4,5] because the anti-symmetric part of the stress tensor and the trace of the couple-stress tensor are not determined by the constitutive equations. These quantities depend on the boundary tractions in specific situations [6]. To overcome these indeterminacies, some alternative theories have been developed. Yang et al. [7] developed a new equilibrium equation to govern the behavior of higher-order stresses by introducing the concept of representative volume and concluded that the couple-stress tensor is symmetric and that the anti-symmetric part of the rotation gradient tensor does not contribute to the deformation energy. This additional equation is named the equilibrium of moments of couples. Hadjesfandiari [8] also developed a modification of the couple-stress theory, in which the couple-stress tensor is skew-symmetric and the skew-symmetric part of the gradient of the rotation tensor is the consistent curvature tensor.

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The first strain gradient of Fleck and Hutchinson [9] includes the effects of gradients of rotation in the equivalent strain rate and is adequate to describe the plastic behavior of materials with internal rotation. A detailed presentation of the strain gradient theories developed by Fleck and Hutchinson can be seen in [10].

The shakedown theory is suitable for the analysis of mechanical components subjected to variable loads during some interval of time, in which only the limits of load variation are known. Polizzotto [11,12] presented a shakedown theory that considers gradient plasticity using the concepts of the residual-based gradient plasticity theory. However, no shakedown theory is available for materials with internal rotation. The purpose of this paper is to present a lower bound shakedown theorem for materials with internal rotation using linear elastic couple-stress theory [1] and the first strain gradient plasticity of Fleck and Hutchinson [9]. An illustrative numerical example of a cantilever beam, with a representative length scale of 6.2 μm, is presented. The beam is subjected to a constant axial load and to an external moment, which varies arbitrarily between prescribed symmetric bounds.

2. Couple-stress theory

The work of the Cosserat brothers [13] was the first to develop a mathematical model to analyze materials with couple-stresses. In the original Cosserat theory, the kinematical quantities were the displacement and microrotation, which is assumed to be independent of the continuum mechanical rotation, frequently called macrorotation. In couple-stress theory, the microrotation is constrained to follow the macrorotation, and the gradient of the rotation is introduced as a second strain measure, named the curvature tensor.

2.1. Kinematics

The kinematic variables in the linear couple-stress theory are the displacement vector u and the rotation vector θ, which is defined as follows:

$$\theta = \frac{1}{2}(\nabla \times u) \tag{1}$$

From these quantities, the following tensors are defined:

- the strain tensor

$$\varepsilon = \mathcal{D}u \tag{2}$$

- the curvature tensor

$$\kappa = \nabla\theta = \mathcal{R}u \tag{3}$$

In Eq. (2), the operator $\mathcal{D}$ represents the linear deformation operator $(1/2)[(\nabla\cdot)+(\nabla\cdot)^T]$ and $\mathcal{R}$ represents the operator defined by $\nabla((1/2)\nabla\times\cdot)$, which relates the curvature tensor κ and the displacement vector u.

2.2. Principle of virtual power

In the linear couple-stress theory, the strain energy density is assumed to be dependent on the strain tensor ε and on the curvature tensor κ [1] and its energetic duals: the symmetric stress tensor σ and the couple-stress tensor μ.

Considering a body that occupies a volume V bounded by a surface S, the expression for the virtual power of the internal forces $\mathcal{P}_i$ [14] is described by

$$\mathcal{P}_i = \langle \sigma, \mathcal{D}\dot{u}\rangle + \langle \mu, \mathcal{R}\dot{u}\rangle \quad \forall \dot{u} \in V \tag{4}$$

where

$$\langle \sigma, \mathcal{D}\dot{u}\rangle = \int_V \sigma : \mathcal{D}\dot{u} : dV \tag{5}$$

$$\langle \mu, \mathcal{R}\dot{u}\rangle = \int_V \mu : \mathcal{R}\dot{u} : dV \tag{6}$$

The expressions for the virtual power of the external field and surface forces and for the virtual power of the external field and surface moments are described by the following linear functionals:

$$L(\dot{u}) = \int_V f\cdot\dot{u}dV + \int_S t\cdot\dot{u}dS \tag{7}$$

$$M(\dot{u}) = \int_V c\cdot\mathcal{H}\dot{u}dV + \int_S m\cdot\mathcal{H}\dot{u}dS \tag{8}$$

where f represents the body force, c represents the body moment, t represents the traction vector, m represents the moment vector, and $\mathcal{H}$ represents the operator $(1/2)(\nabla\times\cdot)$.

The principle of virtual power states that the sum of all internal and external powers must be equal to zero. Therefore

$$\langle \sigma, \mathcal{D}\dot{u}\rangle + \langle \mu, \mathcal{R}\dot{u}\rangle = L(\dot{u}) + M(\dot{u}) \quad \forall \dot{u} \in V \tag{9}$$

According to Eq. (9), a self-equilibrated stress field σ^r can be defined as

$$\langle \sigma^r, \mathcal{D}\dot{u}\rangle = 0 \quad \forall \dot{u} \in V \tag{10}$$

and a self-equilibrated couple-stress field μ^r can be defined as

$$\langle \mu^r, \mathcal{R}\dot{u}\rangle = 0 \quad \forall \dot{u} \in V \tag{11}$$

Assuming that the body forces and moments are equal to zero and neglecting the effects of jump and discontinuities, the equilibrium equation for a linear elastic body [15], is described by

$$\mathcal{D}'\sigma + \mathcal{R}'\mu = 0 \tag{12}$$

subjected to the following boundary conditions

$$t = n\cdot\sigma - \frac{1}{2}n\times\nabla\mu_n \tag{13}$$

$$m = n\cdot\mu - n\cdot\mu_n \tag{14}$$

where n is a unit vector normal to the surface S and μ_n is defined by

$$\mu_n = n\cdot(\mu.n) \tag{15}$$

Because the rotation is not independent of the displacement vector (Eq. (1)), the normal component of the rotation vector is specified by the distribution of the tangential displacements over the boundary [16]. It can then be seen from Eqs. (13) and (14) that only five boundary conditions are necessary.

3. Constitutive equations

Fleck and Hutchinson [9] developed a strain gradient plasticity based on gradients of rotation that fits within the framework of the couple-stress theory. In this theory, the solid is assumed to be incompressible so that only the deviatoric part of the stress tensor and the couple-stress tensor contribute to the strain energy density.

The yield function proposed by [9] defines a bounded region for the plastically admissible stress tensor σ and the couple-stress tensor μ and is given by the following inequality

$$f(\sigma,\mu) = (\sigma_{eq} + l^{-2}\mu_{eq}) - \sigma_Y^2 \le 0 \tag{16}$$