Mechanics Research Communications 62 (2014) 31–36

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

# Thermoelastic damping of micro resonators operating in the longitudinal vibration mode: In comparison with the case of flexural vibration

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ARTICLE INFO

*Article history:*
Received 19 March 2014
Received in revised form 19 August 2014
Accepted 20 August 2014
Available online 28 August 2014

*Keywords:*
Thermoelastic damping
Longitudinal vibration
Micromechanical resonators

ABSTRACT

This paper studies thermoelastic damping in the longitudinal vibration mode. Expression of thermoelastic damping is obtained by using the thermal-energy method and is then validated by comparing with the exact solutions deduced from the coupled thermoelasticity equations. It is demonstrated that Landau–Lifshitz's model overestimates thermoelastic energy loss by employing the adiabatic assumption. Results of the present study indicate that the peak value of thermoelastic damping for isothermal boundary condition is lower than that for adiabatic boundary condition in the longitudinal vibration mode. Furthermore, a comparison was made between longitudinal vibration mode and flexural vibration mode to distinguish their different characteristics. It manifests that thermoelastic damping of rods or beams reaches peak values at the length scale of $10^{-8}$ m for longitudinal vibration in contrast to the order of $10^{-5}$ m or above for flexural vibration in the numerical examples of the present study.



## 1. Introduction

Thermoelastic damping, as an important intrinsic loss mechanism in microelectromechanical systems (MEMS) and nanoelectromechanical systems (NEMS), imposes an upper limit on the quality factor of MEMS or NEMS resonators. The reciprocal of the quality factor (Q-factor), equal to the proportion of energy lost per oscillation in the total mechanical energy in a resonator, is a measure of energy dissipation and Q-factor is a crucial parameter to the sensing performance of MEMS/NEMS resonators based sensors. With extensive applications of MEMS and NEMS technology and growing demand of high precision measurement in biological or chemical detection, resonators with low energy dissipation or high Q-factor are highly desirable.

Zener (1937, 1938), the first scientist proposing the concept of thermoelastic damping, systematically studied the transverse vibration of thin reeds based on so-called "standard anelastic solid model" and gave the well-known approximate formula. Lifshitz and Roukes (2000) studied the thermoelastic damping for flexural vibration of beam resonators of thin rectangular cross section based on one-dimensional model within the context of classical theory of thermoelasticity and they derived explicit expressions for thermoelastic damping and frequency shift. Following Lifshitz and Roukes' study, a great deal of work has been devoted on this issue for different types of resonators ranging from beam, circular plate, rectangular plate to ring, cylinder and hollow cylinder, accounting for either one-dimensional or two-dimensional thermal conduction. These works include, among others, Guo and Rogerson (2003), Sun and Saka (2010), Sharma and Grover (2012), Sharma et al. (2011), Tunvir et al. (2010), Wong et al. (2006), Kim et al. (2010) and Prabhakar and Vengallatore (2008).

Although tremendous amount of study has been conducted on thermoelastic damping of flexural vibration mode, little attention was paid to the thermoelastic damping in longitudinal vibration mode. Stachiv (2013) demonstrated that the attached mass can be measured by detecting the fundamental flexural and longitudinal resonant frequencies. Moreover, the highest mass sensitivity can be achieved in longitudinally vibrating resonators. These findings indicate the importance of longitudinal vibration mode in the mass sensing application.

Landau and Lifshitz (1959) derived thermoelastic damping coefficients for longitudinal vibration of rods and plates under

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http://dx.doi.org/10.1016/j.mechrescom.2014.08.006
0093-6413/

adiabatic assumption, which predict that thermoelastic damping rises with increase of the vibration frequency. Zhang and Turner (2004) presented an approximate method for predicting thermoelastic damping of rod resonators in longitudinal vibration mode. However, the coupled governing equations and the boundary conditions are not accurately satisfied in their method.

The process of thermoelastic damping of longitudinal vibrations is similar to that of flexural vibrations in that energy is dissipated through the irreversible flow of heat from hot regions to cold regions. The difference is that the distance between these regions is dictated by the transverse geometry of the resonator in flexural vibrations while in longitudinal vibrations, hot and cold regions are separated by nodal points in a specific vibration mode along the axial direction of the device. In flexural vibrations, mechanical energy is dissipated mainly by thermal conduction along the thickness direction and temperature distribution in the thickness direction can be readily derived since the elastic dilatation varies linearly along the thickness direction. This characteristic enables one to easily derive an explicit expression of thermoelastic damping as shown in Lifshitz and Roukes (2000). On the contrary, in longitudinal vibration, energy is dissipated by thermal conduction along axial direction and the temperature distribution cannot be solved separately. For this reason, one cannot obtain an explicit expression of thermoelastic damping for longitudinal vibration by using the procedures of Lifshitz and Roukes (2000). Therefore, it is worthwhile to investigate the energy dissipation in the longitudinal vibration mode and explore distinct features in comparison with flexural vibration mode.

In this study, two approaches, thermal-energy method (Hao et al., 2009) and complex-frequency approach, are utilized for analyzing thermoelastic damping of rod resonators operating in longitudinal vibration mode under two boundary conditions: clamped-isothermal, and clamped-adiabatic. Numerical results of thermoelastic damping and frequency shift for longitudinal vibration mode are presented and compared with those of flexural vibration mode. This study is helpful in clarifying the understanding on thermoelastic damping in longitudinal vibrations.

## 2. Thermoelastic damping in the longitudinal vibration

### 2.1. Landau–Lifshitz's model

Landau and Lifshitz (1959) derived expressions for energy dissipation resulting from thermoelastic coupling effect by seeking the dissipated vibration energy, which is equal to the amount of heat flowing from hot regions to cold regions. In their derivation, the adiabatic condition was employed and the temperature gradient is calculated by imposing the adiabatic condition in the coupled equation of thermal conduction. For an isotropic body, the energy loss from vibration energy to thermal energy per cycle of vibration is

$$\Delta Q = \int_0^{2\pi/\omega} \int_v \kappa \frac{\nabla\theta \cdot \nabla\theta}{T_0} dv\, dt, \qquad (1)$$

where $\theta = T - T_0$ is the temperature variation from the initial temperature $T_0$, $\omega$ and $\kappa$ denote the vibration frequency and thermal conductivity, respectively; and $v$ is the volume of the body. For a longitudinally vibrating rod clamped at two ends, with $x$ direction along the rod axis and origin on the left end of the rod, the axial displacement is assumed to have the following form:

$$u_x = u_0 \sin kx \sin \omega t, \qquad (2)$$

where $E_{ad}$ and $u_0$ are the adiabatic Young's modulus and vibration amplitude, respectively; $k = \omega/\sqrt{E_{ad}/\rho}$, and $\rho$ is the mass density. Thus, the stored mechanical energy per cycle of vibration is

$$W = \frac{1}{4} lS\rho u_0^2 \omega^2, \qquad (3)$$

where $S$ and $l$ are the area of cross section and length of the rod, respectively.

By imposing the adiabatic condition in the equation of thermal conduction in the presence of thermoelastic coupling term, the temperature distribution for a longitudinally vibrating rod is obtained as

$$\theta = T - T_0 = -\frac{3T_0\alpha K_{ad}}{\rho C} \nabla \cdot \mathbf{u} = -\frac{T_0 \alpha E_{ad}}{\rho C} k u_0 \cos kx \sin \omega t, \qquad (4)$$

where $K_{ad}$, $\alpha$, $C$ and $\mathbf{u}$ are the adiabatic bulk modulus, linear thermal expansion coefficient, specific heat and elastic displacement vector, respectively.

The quality factor for a micromechanical resonator is defined as

$$Q = 2\pi \frac{W}{\Delta W}, \qquad (5)$$

where $\Delta W$ denotes the energy dissipation per cycle of vibration. For the problem considered in this study the energy dissipation is due to thermoelastic damping effect, thus $\Delta W$ in the above equation equals to $\Delta Q$ given by Eq. (1).

Substituting Eq. (4) into Eq. (1), we obtain $\Delta Q = (\pi\kappa T_0 \alpha^2 E_{ad}^2 / 2\rho^2 C^2 \omega) lSk^4 u_0^2$. Then, the amount of thermoelastic damping expressed in terms of the inverse of the quality factor is obtained as

$$Q^{-1} = \frac{\kappa T_0 \alpha^2 \omega}{\rho C^2}. \qquad (6)$$

### 2.2. Thermal-energy method

A longitudinally vibrating rod of length $l$ is considered. We choose the coordinate system in such a way that the origin is at the left end of the rod and $x$ direction is along the axis of the rod. For this one-dimensional problem, all quantities are only dependent on variable $x$. Within the context of linear theory of thermoelasticity, noting strain components $\varepsilon_x = \partial u_x/\partial x$, $\varepsilon_y = -\nu \partial u_x/\partial x$ and $\varepsilon_z = -\nu \partial u_x/\partial x$ ($\nu$ – Poisson's ratio) in this problem, the governing equations are

$$\begin{cases} \rho \dfrac{\partial^2 u_x}{\partial t^2} = E \dfrac{\partial^2 u_x}{\partial x^2} - E\alpha \dfrac{\partial \theta}{\partial x} \\ \rho C \dfrac{\partial \theta}{\partial t} + E\alpha T_0 \dfrac{\partial^2 u_x}{\partial x \partial t} = \kappa \dfrac{\partial^2 \theta}{\partial x^2} \end{cases}. \qquad (7)$$

To examine the effect of thermoelastic coupling on the longitudinal vibration, we assume that the displacement component $u_x$ and temperature variation $\theta$, are of time-harmonic form:

$$\begin{cases} u_x(x, t) = u(x) e^{i\omega t} \\ \theta(x, t) = \Theta(x) e^{i\omega t} \end{cases}, \qquad (8)$$

where $\omega$ is the angular frequency and $i$ the imaginary unit.

Substitution of Eq. (8) into Eq. (7) gives

$$\begin{cases} -\rho\omega^2 u = E \dfrac{d^2 u}{dx^2} - E\alpha \dfrac{d\Theta}{dx} \\ i\rho C\omega\Theta + iE\alpha T_0 \omega \dfrac{du}{dx} = \kappa \dfrac{d^2\Theta}{dx^2} \end{cases}. \qquad (9)$$

A dimensionless coupling constant $\varepsilon = [E(1+\nu)/(1-\nu)(1-2\nu)](\alpha^2 T_0/\rho C)$ was introduced to represent the strength of thermoelastic coupling (Achenbach, 1984). For common materials used in MEMS/NEMS devices such as silicon