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Induction magnetic stability with a two-component velocity field

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A B S T R A C T

A generalization of the stability results of the induction magnetic field in a simple one space dimensional model for a protoplanetary disc, given in Rüdiger and Shalybkov (2004) and Straughan (2013), is proposed.

The model studied in Rüdiger and Shalybkov (2004) and Straughan (2013) arises from the induction equation for the magnetic field in the presence of a sheared one-component velocity flow with Hall and ion-slip effects. Because of applications in some geophysical problems, here we consider a more general two-component velocity field that generalizes the case studied in Straughan (2013) and also includes elliptic (in particular uniform circular motions $a = -b$) and hyperbolic orbits.

We study linear instability and nonlinear stability of the induction magnetic field by means of the classical spectral and energy methods.

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1. Introduction

The longstanding problem of rapid outward transport of angular momentum mechanism and consequential inward gas spiraling in an accretion disc, had a widely accepted explanation by Balbus and Hawley (1991) which clarified how a local powerful shearing instability, mediated by a weak magnetic field and an outward decreasing angular velocity, should be relevant to understand the origin of the so-called "turbulent viscosity". This instability, known as magnetorotational instability (MRI), requires differentially rotating flows which are ultimately shear flows.

The initial formulation of MRI was for a fully ionized, ideal magnetohydrodynamic medium in which the field and fluid are well coupled to each other. But often non-ideal magnetohydrodynamic effects such as Ohm, Hall and ion-slip effects play an important role. So, in a weakly ionized protoplanetary disk where the electrical conductivity might be too low, the Hall effect can amplify or suppress the MRI. In Rüdiger and Shalybkov (2004), the action of the Hall effect has been considered in a simple 1-D kinematic protoplanetary disc model, referred to a Cartesian frame (x, y, z), where the gas is supposed in motion with velocity $\mathbf{U} = (0, u_0 x, 0)$. The effect was found to be destabilizing when the shear and Hall numbers had opposite signs and unstable perturbations existed, for the same shear number C_Ω connected to u_0, for very small as well as for very high values of the Hall number but only if C_Ω exceeded a minimum value. Recently, Straughan (2013) resumed the 1-D kinematic model of Rüdiger and Shalybkov (2004). He studied the induction magnetic Eq. (1) including also the ion-slip effect and using the same velocity field considered in Rüdiger and Shalybkov (2004). By means of classical variational energy method he found a global exponential stability condition which gives a sharp threshold in correspondence to the minimum value of C_Ω.

Because of some applications to geophysical and astronomical problems (see, for instance, Ponsar et al. (2003) and Hasan et al. (2008)), here we generalize the problem handled in Straughan (2013), by introducing two-components velocity fields. Precisely, we admit that the gas velocity depends not only on the x-variable through the shear coefficient u_0, but also on the y-variable through an other coefficient: $\mathbf{U} = (ay, bx, 0)$. This includes, as special physical cases, the elliptic, circular and hyperbolic motions. Hence we study instability with the standard spectral method and energy nonlinear stability with the Lyapunov second method. Of course, the stability condition we obtain depends on both the parameters a and b of the velocity field. In particular, we find the main result that the most destabilizing case is obtained for the hyperbolic motions (whenever a and b are of the same sign), while the most stabilizing motions are the rotations ($b = -a$), as we expected.

The analysis we perform here is very simple: we use the classical spectral method for studying linear instability and the L_2 energy for nonlinear stability.

The plan of the paper is the following: in Section 2, we recall the main equations; in Section 3 we study linear instability. Section

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4 is devoted to nonlinear energy stability. The paper ends with a conclusion, in Section 5.

2. Magnetic induction equation

The *induction magnetic equation* reads:

$$\frac{\partial B_i}{\partial t} = [\mathrm{curl}(\mathbf{u}\times\mathbf{B})]_i + \eta\Delta B_i - \beta[curl(\mathrm{curl}\mathbf{B}\times\mathbf{B})]_i + \beta_2\left\{\mathrm{curl}[\mathbf{B}\times(\mathbf{B}\times\mathrm{curl}\mathbf{B})]\right\}_i \quad (1)$$

where $i = 1, 2, 3$, $\mathbf{u}$ is an imposed known velocity field, η is the magnetic diffusivity, β is the Hall coefficient and β_2 is the ion-slip coefficient, Cowling (1976).

We shall investigate the stability of the basic solution of (1) $\overline{\mathbf{B}} = (0, 0, B_0)$, with $B_0 \neq 0$.

Here we assume a more general velocity profile, which includes, as a special physical case ($b = -a$), the uniform rotation:

$$\mathbf{u} = (ay, bx, 0),$$

where a and b are two real, nonvanishing, parameters.

In order to study the stability/instability of $\overline{\mathbf{B}}$, we search for a perturbation solution of form

$$\mathbf{B} = (B_0 u(z,t), B_0 v(z,t), B_0).$$

So we obtain the nonlinear system

$$\begin{aligned} u_{,t} &= \left(\frac{1+C_I}{4}\right)u_{,zz} + \hat{C}_\Omega v + \frac{C_H}{4}v_{,zz} \\ &+\frac{C_I}{4}\left[2u(u_{,z})^2 + vu_{,z}v_{,z} + u(v_{,z})^2 + u^2u_{,zz} + uvv_{,zz}\right], \\ v_{,t} &= C_\Omega u + \left(\frac{1+C_I}{4}\right)v_{,zz} - \frac{C_H}{4}u_{,zz} \\ &+\frac{C_I}{4}\left[2v(v_{,z})^2 + uu_{,z}v_{,z} + v(u_{,z})^2 + v^2v_{,zz} + uvu_{,zz}\right], \end{aligned} \quad (2)$$

where

$$C_H = \frac{\beta B_0}{\eta},\; C_\Omega = \frac{bH^2}{\eta},\; \hat{C}_\Omega = \frac{aH^2}{\eta},\; C_I = \frac{B_0^2\beta_2}{\eta}.$$

To system (2) we append the boundary conditions

$$u = 0, v = 0 \quad \text{on} \quad z = 0, 1. \quad (3)$$

The solutions are, for any $t > 0$, elements of an Hilbert space equipped with scalar product, and norm $||\cdot||$. Denoting $U = (u, v)^T$, Eqs. (2) in operator form read

$$U_{,t} = LU + N(U) \quad \text{where} \quad N(0) = 0.$$

L and N are the linear and nonlinear operator respectively, with L given by:

$$L = \begin{pmatrix} \left(\frac{1+C_I}{4}\right)\frac{\partial^2}{\partial z^2} & \hat{C}_\Omega + \frac{C_H}{4}\frac{\partial^2}{\partial z^2} \\ C_\Omega - \frac{C_H}{4}\frac{\partial^2}{\partial z^2} & \left(\frac{1+C_I}{4}\right)\frac{\partial^2}{\partial z^2} \end{pmatrix}.$$

We decompose the operator L into two parts Straughan (2004)

$$L = L_1 + L_2 = \begin{pmatrix} \left(\frac{1+C_I}{4}\right)\frac{\partial^2}{\partial z^2} & \frac{\hat{C}_\Omega + C_\Omega}{2} \\ \frac{\hat{C}_\Omega + C_\Omega}{2} & \left(\frac{1+C_I}{4}\right)\frac{\partial^2}{\partial z^2} \end{pmatrix} + \begin{pmatrix} 0 & \frac{\hat{C}_\Omega - C_\Omega}{2} + \frac{C_H}{4}\frac{\partial^2}{\partial z^2} \\ \frac{C_\Omega - \hat{C}_\Omega}{2} - \frac{C_H}{4}\frac{\partial^2}{\partial z^2} & 0 \end{pmatrix}.$$

The first, L_1, is symmetric, the other one, L_2, is skew-symmetric. Let us define:

$$\mu = \frac{\hat{C}_\Omega - C_\Omega}{2}, \quad \xi = \frac{\hat{C}_\Omega + C_\Omega}{2}.$$

We soon see that the term

$$\frac{\hat{C}_\Omega - C_\Omega}{2} + \frac{C_H}{4}\frac{\partial^2}{\partial z^2}$$

is stabilizing (because it is a skew-symmetric term), moreover also the ion-slip effect is stabilizing.

As stability parameter we choose

$$\xi^2 = \frac{(\hat{C}_\Omega + C_\Omega)^2}{4}$$

(see Galdi and Straughan (1985) and Mulone and Rionero (2003) for the stability parameter choice in the case of the rotating or magnetic Bénard problem.) We note that in the special physical case of the uniform rotation, we have $\xi = 0$ and the basic magnetic field is stable (see below).

3. Linear instability

In order to study linear instability, we consider the system

$$\begin{aligned} u_{,t} &= \left(\frac{1+C_I}{4}\right)u_{,zz} + \hat{C}_\Omega v + \frac{C_H}{4}v_{,zz} \\ v_{,t} &= C_\Omega u + \left(\frac{1+C_I}{4}\right)v_{,zz} - \frac{C_H}{4}u_{,zz}. \end{aligned} \quad (4)$$

System (4) is linear and autonomous. Because of the boundary conditions, as usual, we look for solutions of the form

$$u = Ue^{-\sigma_n t}\sin n\pi z, \quad v = Ve^{-\sigma_n t}\sin n\pi z \quad V, U \in \mathbb{R},$$

where σ_n is a priori a complex number. Then, σ_n has to satisfy the equation

$$\left(\sigma_n - \frac{1+C_I}{4}n^2\pi^2\right)^2 - \left(\xi - \mu + \frac{C_H}{4}n^2\pi^2\right)\left(\xi + \mu - \frac{C_H}{4}n^2\pi^2\right) = 0. \quad (5)$$

In the particular case $\xi = 0$, we have

$$\left(\sigma_n - \frac{1+C_I}{4}n^2\pi^2\right)^2 + \left(\mu - \frac{C_H}{4}n^2\pi^2\right)^2 = 0. \quad (6)$$

The last equation immediately implies the following Theorem:

Theorem 3.1. *By assuming that* $\mathbf{u}$ *is a uniform rotation vector, i.e.* $\mathbf{u} = (ay, -ax, 0)$, *then the basic magnetic field* $\overline{\mathbf{B}} = (0, 0, B_0)$ *is linearly stable, i.e.,* $re(\sigma_n) > 0$ *for any* $n = 1, 2, \ldots$