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On the unlimited gain of a nonlinear parametric amplifier

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ABSTRACT

The present paper is concerned with analysis of the response of a nonlinear parametric amplifier in a broad range of system parameters, particularly beyond resonance. Such analysis is of particular interest for micro- and nanosystems, since many small-scale parametric amplifiers exhibit a distinctly nonlinear behavior when amplitude of their response is sufficiently large. The modified method of direct separation of motions is employed to study the considered system. As the result it is obtained that steady-state amplitude of the nonlinear parametric amplifier response can reach large values in the case of arbitrarily small amplitude of external excitation, so that the amplifier gain tends to infinity. Very large amplifier gain can be achieved in a broad range of system parameters, in particular when the amplitude of parametric excitation is comparatively small. The obtained results clearly demonstrate that very meaningful parametric amplification can be realized in resonant systems driven within a nonlinear response regime, and that nonlinear parametric amplifier possesses certain advantages over linear one.

1. Introduction

Parametric amplifiers based on resonant micro- and nanosystems appeared to be an efficient tool for low-noise, low-distortion signal amplification, and recently have received significant attention in applied physics and engineering research communities (Rugar and Grutter, 1991; Dana et al., 1998; Krylov et al., 2010; Karabalin et al., 2010). Conventional parametric amplifiers are linear systems under action of combined external and parametric excitation. Dynamics of such systems is thoroughly studied (see, e.g. (Yakubovich and Starzhinskii, 1975)). However, many small-scale parametric amplifiers based on micro- and nanosystems exhibit a distinctly nonlinear behavior when amplitude of their response is sufficiently large (Postma et al., 2005; Lifshitz and Cross, 2008). So, it becomes necessary to realize such systems dynamics in a nonlinear context. The Duffing-type nonlinearity can be considered as the simplest model. For example, in paper (Rhoads and Shaw, 2010) the near-resonant response of such system was studied for small values of parametric excitation amplitude and nonlinearity coefficient. In the present paper this system is considered in a broader range of parameters. We abandon the requirement for the natural frequency of the corresponding linearized system to be close to the external excitation frequency, and do not consider the parametric excitation amplitude and the nonlinearity coefficient as necessarily small. So, the following equation is studied:

$$z'' + \gamma z' + \delta z + \chi z \cos 2t_0 + kz^3 = A\cos(t_0 + \phi) \qquad (1)$$

Here z represents the amplifier response, γ is the coefficient of dissipation, which is assumed to be linear, A and χ the amplitudes of the external and parametric excitations correspondingly, ϕ the relative phase term, δ the squared natural frequency of the linearized system, and t_0 the dimensionless time.

For studying Eq. (1) we employ the modified method of direct separation of motions (MDSM) (Sorokin, 2014). The conventional MDSM is a method which facilitates solution of various problems involving action of high-frequency vibrations on nonlinear mechanical systems (see, e.g. (Blekhman, 2000, 2004)). The modified MDSM is adapted for analysis of linear and nonlinear differential equations without explicit small parameter. A general comparison of this method with other approaches, particularly Ritz's method of harmonic balance (Chelomey, 1979; Magnus, 1965), Van der Pol's method of slowly varying amplitudes (Magnus, 1965; Nayfeh, 2000; Krylov and Bogoliubov, 1947), Krylov–Bogoliubov–Mitropolsky methods (Nayfeh, 2000; Krylov and Bogoliubov, 1947; Bogoliubov and Mitropolskii, 1961; Sanders and Verhulst, 1985; Nayfeh and Mook, 1979; Nayfeh, 2005), and the multiple scales method (Nayfeh, 2000; Nayfeh and Mook, 1979;

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Nayfeh, 2005), is given in paper (Sorokin, 2014). In particular, it is noted that the modified MDSM may be interpreted as a certain continuation of the averaging methods.

Widely used approaches for analysis of systems with periodic coefficients are based on Floquet theory (Brillouin, 1953). However, application these approaches for strongly nonlinear problems is rather problematic.

2. Solution by the modified method of direct separation of motions

Employing the modified MDSM for solving Eq. (1), we search solution in the form:

$$z = \alpha(T_1) + \psi(T_1, T_0) \tag{2}$$

Where new timescales T_1 and T_0 are defined as $T_0 = t_0$, $T_1 = \varepsilon T_0$; $\varepsilon \ll 1$ is an artificial small parameter, α is "slow", and ψ is "fast", 2π-periodic in dimensionless time T_0 variable, with average zero:

$$\left\langle \psi(T_1, T_0) \right\rangle = 0$$

Here $\langle\ldots\rangle$ designates averaging in the period 2π on time T_0, i.e. for function $h(T_1,T_0)$ we have $\left\langle h(T_1, T_0)\right\rangle = 1/2\pi \int_0^{2\pi} h(T_1, T_0) dT_0$.

The method may be interpreted in the following way: It is applicable for solving equations without small parameter; however, there is a fee one has to pay for this. The fee lies in the imposition of a certain restriction on the sought solutions: only solutions which are close to periodic may be determined by the means of the method (so, we assume that the considered equation has periodic solutions, which we also detect by the MDSM). The restriction expresses itself in the introduction of the artificial small parameter ε, which defines the proximity of the sought solution to a periodic one. When this parameter tends to zero, we get pure periodic in time t_0 solution.

Similarly to the multiple scales method (Nayfeh, 2000; Nayfeh and Mook, 1979), the modified MDSM implies variables T_1 and T_0 to be considered independent, so that $d^2/dt_0^2 = \partial^2/\partial T_0^2 + 2\varepsilon(\partial^2/\partial T_1 \partial T_0) + \varepsilon^2(\partial^2/\partial T_1^2)$. Inserting (2) and timescales T_1 and T_0 into (1) and averaging this equation on time T_0 with the conditions for ψ being taken into account, we obtain the following equation of "slow" motion (for variable α)

$$\varepsilon^2 \frac{d^2\alpha}{dT_1^2} + \varepsilon\gamma \frac{d\alpha}{dT_1} + \delta\alpha + \chi\langle\psi \cos 2T_0\rangle + k(\alpha^3 + 3\alpha\langle\psi^2\rangle + \langle\psi^3\rangle) = 0, \tag{3}$$

Equation of "fast" motions (for variable ψ) can be obtained by subtracting Eq. (3) from Eq. (1)

$$\frac{\partial^2\psi}{\partial T_0^2} + 2\varepsilon \frac{\partial^2\psi}{\partial T_1 \partial T_0} + \varepsilon^2 \frac{\partial^2\psi}{\partial T_1^2} + \gamma\left(\frac{\partial\psi}{\partial T_0} + \varepsilon\frac{\partial\psi}{\partial T_1}\right) + k(\psi^3 + 3\alpha\psi^2 + 3\alpha^2\psi - 3\alpha\langle\psi^2\rangle - \langle\psi^3\rangle) + \delta\psi = -\chi((\alpha+\psi)\cos 2T_0 - \langle\psi\cos 2T_0\rangle) + A\cos(T_0 + \phi) \tag{4}$$

Taking into account that $\psi(T_1, T_0)$ is a time T_0 periodic function, solution of the fast motions Eq. (4) is sought in the form of series

$$\psi = B_1(T_1)\cos(T_0 + \theta_1(T_1)) + B_2(T_1)\cos(2T_0 + \theta_2(T_1)) + \ldots \tag{5}$$

Influence of the second and the third harmonics (and all higher harmonics) on the system response for $\delta = O(1)$ and $\chi = O(1)$ turns out to be negligibly weak when either the nonlinearity coefficient k or the external excitation amplitude A is small: $k \ll 1$ or $A \ll 1$. In particular, no super- or sub-harmonic resonances can occur. So, in this range of parameters only the first harmonic can be taken into account to predict the system response. Accounting of other harmonics is not difficult, but leads only to a minor quantitative change of the results. For amplitude B_1 and phase θ_1 the following equations are obtained:

$$\varepsilon^2 \frac{d^2B_1}{dT_1^2} + \varepsilon\gamma\frac{dB_1}{dT_1} - B_1\left(1 + \varepsilon\frac{d\theta_1}{dT_1}\right)^2 + \delta B_1 + \frac{3}{4}kB_1^3 + 3k\alpha^2 B_1 = -\frac{1}{2}\chi B_1 \cos 2\theta_1 + A\cos(\theta_1 - \phi) \tag{6}$$

$$\varepsilon^2 B_1 \frac{d^2\theta_1}{dT_1^2} + \left(\gamma B_1 + 2\varepsilon\frac{dB_1}{dT_1}\right)\left(1 + \varepsilon\frac{d\theta_1}{dT_1}\right) = \frac{1}{2}\chi B_1 \sin 2\theta_1 - A\sin(\theta_1 - \phi), \tag{7}$$

The stable steady-state response of the amplifier is of primary interest. So the following system of equations is composed to describe it:

$$\delta\alpha + k\left(\alpha^3 + 3\alpha\frac{B_1^2}{2}\right) = 0, \tag{8}$$

$$-B_1 + \delta B_1 + \frac{3}{4}kB_1^3 + 3k\alpha^2 B_1 = -\frac{1}{2}\chi B_1 \cos 2\theta_1 + A\cos(\theta_1 - \phi) \tag{9}$$

$$\gamma B_1 = \frac{1}{2}\chi B_1 \sin 2\theta_1 - A\sin(\theta_1 - \phi), \tag{10}$$

In the present paper we consider relations $\delta > 0$, $k > 0$ as fulfilled, so Eq. (8) has single real solution $\alpha = 0$. From the derived equation of slow motion (3) it follows that this solution is always stable.

3. Negligible small amplitude of external excitation

First, examine the case of negligible small amplitude of external excitation $A \sim \varepsilon^2$. Taking into account that $\alpha = 0$, from Eqs. (9) and (10) obtain the following expressions for amplitude B_1:

$$B_1 = 0,\; B_1 = \sqrt{\frac{4}{3k}\left(\pm\sqrt{\frac{1}{4}\chi^2 - \gamma^2} + (1-\delta)\right)} \tag{11}$$

So, when relations $\chi > 2\gamma$, $\sqrt{1/4\chi^2 - \gamma^2} + 1 - \delta > 0$ hold true, stable oscillations with amplitude $B_1 = \sqrt{4/(3k)(\sqrt{1/4\chi^2 - \gamma^2} + (1-\delta))}$ can arise in the considered system even for arbitrarily small value of the external excitation amplitude A. Stability of these oscillations follows from Eqs. (6) and (7). As an illustration, the dependencies of the steady-state amplitude B_1 of the nonlinear parametric amplifier response on parameter δ are shown in Fig. 1. Solid lines correspond to stable branches, and dashed lines to unstable branches.

From expressions (11) and Fig. 1 it follows, in particular, that large amplifier response can be obtained when the parametric excitation amplitude χ is comparatively small: $\chi \sim \varepsilon$. For example, when relations $\chi \sim \varepsilon$, $\chi > 2\gamma$, $\delta \not\approx 1$ and $k \sim \varepsilon$ hold true, expression for B_1 takes the form:

$$B_1 = \sqrt{\frac{4}{3k}(1-\delta)}, \tag{12}$$

and when $\delta < 1$, we get $B_1 \sim \varepsilon^{-1/2}$. As an illustration, the amplifier response is shown in Fig. 2 for $\chi = 0.1$, $k = 0.001$, $\gamma = 0.02$, $A = 0.00001$ and (a) $\delta = 0.96$, $\dot{z}(0) = 0$, $z(0) = 0$, (b) $\delta = 0.5$, $\dot{z}(0) = 0$, $z(0) = 27$. Nonzero initial conditions are imposed in case (b) in order to get in basin of attraction of the required regime of steady-state oscillations with large amplitude (two stable regimes coexist at these values of the parameters, see Fig. 1). Here solid lines correspond to the numerical solution of the initial equation (Wolfram Mathematica, NDSolve), and dashed lines designate value of the amplitude of