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Numerical analysis of wrinkled, anisotropic, nonlinearly elastic membranes

A. Atai [a], D.J. Steigmann [b,*]

[a] Department of Mechanical Engineering, University of Tehran, Tehran, Iran
[b] Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA

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ABSTRACT

The method of dynamic relaxation is used to simulate wrinkling in anisotropic sheets modelling bio-tissues and structural membranes.

1. Introduction

In this work we extend the dynamic relaxation method, applied to the study of wrinkling in isotropic membranes in Haseganu and Steigmann (1994), to the analysis of partly wrinkled *anisotropic* membranes. The considered model is based on *tension-field theory* (Pipkin, 1994), which is derived from conventional membrane theory via quasiconvexification of the associated strain-energy function (Dacarogna, 1982), yielding the minimum strain energy that can be attributed to a given state of strain. The relaxed strain-energy function automatically excludes the destabilizing compressive stresses predicted by the original, unrelaxed, energy. It is derived by constructing an energy-minimizing sequence of deformations characterized by ever-more finely spaced wrinkles and effectively encodes the average energy density of a membrane containing many wrinkles. The same model emerges rigorously from three-dimensional nonlinear elasticity via the method of gamma convergence (Le Dret and Raoult, 1995), a technique for extracting the leading-order variational problem in the small-thickness limit.

Whereas the expression for the potential energy in tension-field theory has been rigorously justified, the question of the existence of energy minimizers remains open. This is due to the failure of the relaxed strain-energy function to satisfy coercivity conditions underlying the hypotheses of available existence theorems based on the direct method of the calculus of variations (Dacarogna, 1982). This circumstance stems from the fact that the absence of stress in connected regions of strain space – a feature of the relaxed energy – implies an absence of stiffness also. This situation furnishes impetus for methods such as that adopted here, in which the equilibrium problem is embedded in an artificial dynamical system constructed in such a way that the desired equilibria are globally asymptotically stable with respect to arbitrary initial data. Indeed, such an approach may prove fruitful in devising constructive existence theorems; however, this issue is beyond our present scope.

We discretize the surrogate dynamical system using a finite-difference method based on Green's theorem, and forward integrate in (artificial) time using a simple difference scheme. Temporal accuracy is not an issue, as it is only the asymptotic states that are of interest. This allows for the use of simple explicit finite difference operators to achieve an efficient vectorized system for computations.

The basic theory and associated numerical analysis are well developed, and discussed in detail in the references cited. For this reason we forego their detailed description, emphasizing instead their application to anisotropic bio-elastic or artificial structural membranes. The strain-energy function adopted is inspired by the observed response of bio-tissues, in which an initially soft in-plane tensile response is followed by strain-stiffening as the underlying collagen fibers straighten and stretch on the micro-scale to accommodate an overall finite deformation. A similar mechanism characterizes woven PVC-clad polyester structural fabric used in tension structures, in which the interlaced fibers of the weave are initially crimped and hence relatively soft in response to in-plane tension, stiffening as the fibers straighten in the transition to a stretching mode (Nadler et al., 2006). Alternative analyses of

* Corresponding author. Tel.: +1 510 684 5380; fax: +1 510 642 6144.
E-mail address: dsteigmann@berkeley.edu (D.J. Steigmann).

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wrinkling in anisotropic membranes are discussed in Woo et al. (2004), Barsotti and Vannucci (2013) for the case of small strains. In the present work we model finite strains, using Pipkin's algorithm (Pipkin, 1994) to construct relaxed membrane theory. This is applicable if the underlying (unrelaxed) energy is a convex function of strain – a condition that is usually satisfied in applications. An interesting alternative treatment is described in Epstein (1999).

2. Membrane theory

We have provided a brief resumé of the theoretical background in a companion paper (Atai and Steigmann, 2012). This is summarized here for the benefit of the reader. The equilibrium equation for membranes has a simple divergence structure identical to that for conventional bulk continua. We use a plane Ω as reference configuration, taken here to be unstressed. Let $\mathbf{e}_\alpha$; $\alpha = 1, 2$, be orthonormal vectors spanning (the translation space of) Ω, and let $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. We use the usual summation convention with Greek indices taking values in {1, 2} and Latin indices in {1, 2, 3}, while Greek subscripts preceded by commas are used to denote partial derivatives with respect to initial Cartesian coordinates x_α. The equilibrium equation in the absence of lateral loads is

$$div\mathbf{T} = \mathbf{0}, \quad \text{or} \quad T_{i\alpha,\alpha} = 0, \tag{1}$$

where

$$\mathbf{T} = T_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha \tag{2}$$

is the Piola stress (resultant) and *div* is the two-dimensional divergence operator on Ω. In the case of elasticity the stress is determined by the deformation gradient

$$\mathbf{F} = F_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha, \quad \text{where} \quad F_{i\alpha} = r_{i,\alpha} \equiv \frac{\partial r_i}{\partial x_\alpha} \tag{3}$$

in which r_i are the Cartesian coordinates of a material point after deformation. The relationship is (Haseganu and Steigmann, 1994)

$$\mathbf{T} = W_{\mathbf{F}}, \quad \text{or} \quad T_{i\alpha} = \frac{\partial W}{\partial F_{i\alpha}}, \tag{4}$$

where W is the strain energy per unit area of Ω.

These equations are augmented by traction data $t_i = T_{i\alpha}\nu_\alpha$ or position data $r_i = R_i$ on complementary parts of the boundary, where ν_α are the components of the exterior unit normal to an edge in the reference configuration.

For the mixed zero-load/position boundary-value problems considered here, equilibria furnish energy minimizers only of their gradients $\mathbf{F}$ satisfy the Legendre-Hadamard condition

$$\mathbf{a}\otimes\mathbf{b}\cdot\mathcal{M}(\mathbf{F})[\mathbf{a}\otimes\mathbf{b}] \geq 0, \tag{5}$$

pointwise in Ω, for all non-zero three-vectors $\mathbf{a}$ and two-vectors $\mathbf{b} \in \Omega$, where

$$\mathcal{M}(\mathbf{F}) = W_{\mathbf{FF}} \tag{6}$$

is the tensor of elastic moduli. We shall also make use of the strain-dependent elastic moduli $\mathcal{C}(\mathbf{E})$, where

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^t\mathbf{F} - \mathbf{1}); \quad E_{\alpha\beta} = \frac{1}{2}(F_{i\alpha}F_{i\beta} - \delta_{\alpha\beta}) \tag{7}$$

is the strain in which $\mathbf{1}$ is the identity for 2-space, and

$$\mathcal{C}(\mathbf{E}) = U_{\mathbf{EE}} \tag{8}$$

are the plane-stress elastic moduli, in which

$$U(\mathbf{E}) = W(\mathbf{F}) \tag{9}$$

is the associated strain-energy function.

An application of the chain rule furnishes the useful connection

$$\mathcal{M}(\mathbf{F})[\mathbf{A}] = \mathbf{AS} + \frac{1}{2}\mathbf{F}\mathcal{C}(\mathbf{E})[\mathbf{A}^t\mathbf{F} + \mathbf{F}^t\mathbf{A}] \tag{10}$$

for any tensor $\mathbf{A}$ of the form $\mathbf{A} = A_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha$, where

$$\mathbf{S} = U_{\mathbf{E}} \tag{11}$$

is the symmetric (plane) second Piola-Kirchhoff stress ($\mathbf{S} = S_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$), related to the Piola stress by

$$\mathbf{T} = \mathbf{FS}; \quad T_{i\alpha} = F_{i\beta}S_{\beta\alpha}. \tag{12}$$

We observe, using the minor symmetries of $\mathcal{C}$, that

$$\mathbf{A}\cdot\mathcal{M}(\mathbf{F})[\mathbf{A}] = \mathbf{A}^t\mathbf{A}\cdot\mathbf{S} + \mathbf{A}^t\mathbf{F}\cdot\mathcal{C}(\mathbf{E})[\mathbf{A}^t\mathbf{F}]. \tag{13}$$

In terms of components,

$$\mathcal{M}_{i\alpha j\beta}A_{i\alpha}A_{j\beta} = A_{i\alpha}A_{i\beta}S_{\beta\alpha} + \mathcal{C}_{\alpha\beta\lambda\mu}A_{i\alpha}F_{i\beta}A_{j\lambda}F_{j\mu}. \tag{14}$$

The strain-energy function W is locally convex (as a function of $\mathbf{F}$) if and only if $\mathbf{A}\cdot\mathcal{M}(\mathbf{F})[\mathbf{A}] \geq 0$ for all $\mathbf{A}$. Similarly, the strain-energy function U is locally convex (as a function of $\mathbf{E}$) if and only if $\mathbf{B}\cdot\mathcal{C}(\mathbf{E})[\mathbf{B}] \geq 0$ for all $\mathbf{B}$. It follows that if U is convex in $\mathbf{E}$ and the stress is positive semi-definite, then W is convex. This can be seen immediately by using the spectral representation $\mathbf{S} = \sum_{\alpha=1}^{2} S_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha$, where S_α are the principal stresses and $\mathbf{u}_\alpha$ are the associated principal directions; thus, if the principal stresses are non-negative, then

$$\mathbf{A}^t\mathbf{A}\cdot\mathbf{S} = \sum_{\alpha=1}^{2} S_\alpha \left|\mathbf{A}\mathbf{u}_\alpha\right|^2 \geq 0 \tag{15}$$

for all non-zero $\mathbf{A}$. In the present context this observation is due to Pipkin (1993).

In fact it is well known (Haseganu and Steigmann, 1994) in membrane theory that positive semi-definiteness of the 2nd Piola-Kirchhoff stress is a *necessary* condition for the Legendre-Hadamard inequality and hence necessary for the energy to be minimized. This observation is the basis of tension-field theory, in which the membrane strain-energy function is relaxed so as to ensure that a negative-definite or indefinite stress never arises in any configuration (Pipkin, 1994).

3. Solution procedure

We use a discrete version of the Green-Stokes theorem to discretize the equations directly on the computational plane Ω. This method, incorporating position and traction boundary conditions, is described comprehensively by Haseganu and Steigmann (1994), to which reference may be made for a detailed discussion. The associated nodal points are distributed at the intersections of a curvilinear grid of boundary-fitted coordinates computed using the grid-mapping procedure discussed in (Wang and Steigmann, 1997) and so again we omit the details.

This scheme is used to discretize the *artificial* dynamical system

$$div\mathbf{T} = \rho\ddot{\mathbf{r}} + c\dot{\mathbf{r}}, \tag{16}$$

obtained by appending an explicit viscous damping term to the inertial term in the actual equations of motion. In turn, the spatially discretized system is used together with an explicit central-difference time integration scheme designed for efficient vectorization of the equations. This is *not* the actual equation of motion. It is an artificial system introduced solely to expedite the computation of equilibria.