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New solvable problems in the dynamics of a rigid body about a fixed point in a potential field

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ABSTRACT

We determine the general form of the potential of the problem of motion of a rigid body about a fixed point, which allows the angular velocity to remain permanently in a principal plane of inertia of the body. Explicit solution of the problem of motion is reduced to inversion of a single integral. A several-parameter generalization of the classical case due to Bobylev and Steklov is found. Special cases solvable in elliptic and ultraelliptic functions of time are discussed.

1. Introduction

1.1. Historical

Integrable systems constitute a rare exception in Hamiltonian mechanics. This is most clearly manifested in the field of rigid body dynamics in various problem settings, where all known general integrable cases (integrable for arbitrary initial motion) constitute few small tables, see e.g. Leimanis (1965), Yehia (1999) and Borisov and Mamaev (2005). For some new solutions and few recently added integrable cases see Borisov et al. (2008), Yehia and El-Mandouh (2013, 2011, 2008), Yehia (2012). In the classical problem of motion of a rigid body about a fixed point in a uniform gravity field, integrable only are the three famous cases named after Euler, Lagrange and Kowalevski (Kovalevskaya), see e.g. Leimanis (1965). The problem of the heavy gyrostat, resulting from the former by the addition of a symmetric rotor with its axis fixed in the body, has three integrable cases generalizing the classical three cases. Those are the case of Joukovsky, the case of axially symmetric gyrostat known as Lagrange's case (Leimanis, 1965) and the case due to Yehia (1986).

Another type of problems is named conditionally integrable. Those are integrable only on a fixed level of the areas integral, usually the zero level. For them the procedure of integration of Hamiltonian systems applies on that level, where the problem degrees of freedom are lowered by one. A classical example of such cases is the one due to Goriachev and Chaplygin and its generalization to the gyrostat e.g. Leimanis (1965). A relatively large number of those cases is found recently, which are mainly generalizations of Kowalevski's case and like it admit an integral quartic in velocities, but differs in that this integral is conditional, valid only on the zero level of the areas integral. At present we have eight cases of this type, with potential (and gyroscopic in cases) forces (Yehia, 1999, 2006, 2012; Yehia and El-Mandouh, 2013, 2011, 2008). This relative abundance is a result of the use of new a method of construction of integrable 2D Lagrangian systems, of which integrable rigid body dynamics come out as special cases e.g. Yehia (2006). Nevertheless, we do not know whether there are more integrable potentials for a body with the Kowalevski configuration, and in that case how many and how to find them?

Second to integrable cases comes particular solutions of equations of motion of a rigid body in various settings. Those are solutions subject to certain conditions on the position and angular velocity, not only on the integrals of motion. For such problems, the phase space does not necessarily have Hamiltonian structure and usually one has to manage the equations of motion and the conditions and to find a suitable way for performing separation of variables.

In the present note we give a generalization of one of the above particular solutions, namely the solution due to Bobylev and

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Steklov. We determine the general form of the potential for which the Bobylev–Steklov condition on the angular velocity is satisfied. The angular velocity stays permanently in one of the principal planes of inertia of the body at the fixed point. The new solution is valid for an arbitrary rigid body without the restriction of the Bobylev–Steklov case on the moments of inertia.

1.2. Formulation of the problem

Assume that the body is acted upon by certain potential forces, which admit a symmetry axis fixed in space. The equations of motion for this problem can be written in the Euler–Poisson form (e.g. (Leimanis, 1965)):

$$\begin{aligned} A\dot{p}+(C-B)qr &= \gamma_2\frac{\partial V}{\partial\gamma_3}-\gamma_3\frac{\partial V}{\partial\gamma_2},\\ B\dot{q}+(A-C)pr &= \gamma_3\frac{\partial V}{\partial\gamma_1}-\gamma_1\frac{\partial V}{\partial\gamma_3},\\ C\dot{r}+(B-A)pq &= \gamma_1\frac{\partial V}{\partial\gamma_2}-\gamma_2\frac{\partial V}{\partial\gamma_1}, \end{aligned} \quad (1)$$

$$\dot{\gamma}_1+q\gamma_3-r\gamma_2=0,\quad \dot{\gamma}_2+r\gamma_1-p\gamma_3=0,\quad \dot{\gamma}_3+p\gamma_2-q\gamma_1=0, \quad (2)$$

where A, B, C are the principal moments of inertia, p, q, r are the components of the angular velocity of the body and γ_1, γ_2, γ_3 are the components of the unit vector γ along the axis of symmetry of the force field, all being referred to the principal axes of inertia at the fixed point. The potential V depends only on the Poisson variables $\gamma_1, \gamma_2, \gamma_3$. In the classical problem of a heavy body $V=a\gamma_1+b\gamma_2+c\gamma_3$.

Eqs. (1) and (2) admit three general first integrals:

$$I_1=\frac{1}{2}Ap^2+\frac{1}{2}Bq^2+\frac{1}{2}Cr^2+V,\ \text{the energy integral} \quad (3)$$

$$I_2=\gamma_1^2+\gamma_2^2+\gamma_3^2=1,\ \text{the geometric integral} \quad (4)$$

$$I_3=Ap\gamma_1+Bq\gamma_2+Cr\gamma_3,\ \text{the areas integral} \quad (5)$$

In the classical problem of motion of a rigid body about a fixed point in a uniform gravity field there are eleven solutions of this type known after authors of the 19th and the 20th centuries. All of them are collected in Table 1 below (in chronological order):

For a detailed account of those cases see Gorr (2010) or Dokshevich (1992). Some of them were generalized through the addition of a gyrostatic moment (Gorr, 2010) and other potential and gyroscopic forces (Gorr, 2010; Yehia, 1988).

In the present article we aim at exploring the possibility of particular solutions of the Bobylev–Steklov type for the problem of motion of a rigid body about a fixed point in a field that generalizes the classical setting.

2. A new solvable case

The aim of this paper is to look for potentials V which allow full solution of the system (1) and (2) under the condition

$$q=0 \quad (6)$$

on the zero level of the areas integral, i.e.

$$I_3=0 \quad (7)$$

We assume that $A\neq C$, without restriction on the third moment of inertia B. This choice of the problem is motivated by the classical Bobylev–Steklov solution, which is characterized by the same condition (6) but with the potential $V=a\gamma_1$ and the additional restriction on the moments of inertia $A=2C$, which is not imposed here.

Our result is formulated in the following

Theorem 1. *For an arbitrary rigid body moving about a fixed point while acted upon by forces with a potential V satisfying the linear partial differential equation*

$$\begin{aligned} &\gamma_1\gamma_2\gamma_3\left(A\frac{\partial^2 V}{\partial\gamma_1^2}-C\frac{\partial^2 V}{\partial\gamma_3^2}\right)-\gamma_2(A\gamma_1^2-C\gamma_3^2)\frac{\partial^2 V}{\partial\gamma_1\partial\gamma_3}+(A\gamma_1^2\\ &+C\gamma_3^2)\left(\gamma_1\frac{\partial^2 V}{\partial\gamma_2\partial\gamma_3}-\gamma_3\frac{\partial^2 V}{\partial\gamma_1\partial\gamma_2}\right)-(A-2C)\gamma_2\gamma_3\frac{\partial V}{\partial\gamma_1}\\ &+2(A-C)\gamma_3\gamma_1\frac{\partial V}{\partial\gamma_2}-(2A-C)\gamma_1\gamma_2\frac{\partial V}{\partial\gamma_3}=0 \end{aligned} \quad (8)$$

the Euler–Poisson equations (1) and (2) admit the solution parametrized in terms of γ_1 by expressions

$$\begin{aligned} p &= -\sqrt{\frac{C\gamma_3}{A(A-C)\gamma_1}\left(\gamma_1\frac{\partial V}{\partial\gamma_3}-\gamma_3\frac{\partial V}{\partial\gamma_1}\right)},\\ q &= 0,\\ r &= \sqrt{\frac{A\gamma_1}{C(A-C)\gamma_3}\left(\gamma_1\frac{\partial V}{\partial\gamma_3}-\gamma_3\frac{\partial V}{\partial\gamma_1}\right)} \end{aligned} \quad (9)$$

$$\gamma_3=\lambda\gamma_1^{C/A},\qquad \lambda=\text{const.} \quad (10)$$

$$\gamma_2=\sqrt{1-\gamma_1^2-\lambda^2\gamma_1^{(2C/A)}} \quad (11)$$

and the relation with time is given by

$$t=\int\frac{d\gamma_1}{\sqrt{g(\gamma_1)}} \quad (12)$$

where

$$g(\gamma_1)=\frac{A\gamma_1^{(A-C)/A}}{C(A-C)\lambda}(1-\gamma_1^2-\lambda^2\gamma_1^{(2C/A)})\left(\gamma_1\frac{\partial V}{\partial\gamma_3}-\gamma_3\frac{\partial V}{\partial\gamma_1}\right)_0$$

and $()_0$ in the right hand side means the value of the expression in virtue of the relations (10) and (11), so that g is a function of the single variable γ_1.

Proof. From (6) and the middle equation of (1) we get

$$(A-C)pr-\gamma_3\frac{\partial V}{\partial\gamma_1}+\gamma_1\frac{\partial V}{\partial\gamma_3}=0 \quad (13)$$

Differentiating this equality and using Eqs. (1)–(7) we arrive at two homogeneous linear algebraic equations whose compatibility condition gives Eq. (8) for V. Using one of them with Eq. (13) we get the expressions (9) for p, q.

On the other hand, from the first and third equations of (2), in virtue of (6), we have

$$\frac{d\gamma_3}{d\gamma_1}=\frac{\dot{\gamma}_3}{\dot{\gamma}_1}=\frac{C}{A}\frac{\gamma_3}{\gamma_1} \quad (14)$$

This can be readily integrated to give (10), and then (11) follows from (4).

Thus, five of the six Euler–Poisson variables are expressed in terms of γ_1. The relation (12) with time can be determined by separation of variables in the first of the set of Poisson equations.□

3. The general form of the solution

It is not hard to construct the general solution of the linear PDE (8), which may be written in the form

$$V=V_1+V_2, \quad (15)$$