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Functionally graded hollow cylinders with arbitrary varying material properties under nonaxisymmetric loads

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ABSTRACT

A functionally graded circular hollow cylinder is studied analytically under arbitrarily non-uniform loads on the inner and outer surfaces. The elastic properties are assumed to vary arbitrarily through the thickness. By dividing the cylinder into some homogeneous sub-cylinders, the analytical solutions for the stresses and displacements are derived explicitly. Some numerical results, based on the presented analytical solutions, are provided to investigate the effects of material non-homogeneity on the distribution of the stresses under the non-uniform internal load. The results show that the behavior of the functionally graded cylinder under the nonaxisymmetric load has some special characteristics other than that of the uniformly pressurized one.

1. Introduction

In recent years, developments in the aerospace engineering, nuclear power plants, biochemical engineering etc. have raised new demands on the materials which should have different properties to suit the unusual service conditions. As new advanced composites, functionally graded materials (FGMs) have some excellent thermo-mechanical properties, such as high strength, superb heat resistance, which the common materials do not possess simultaneously. Thus, many researchers have devoted their attention to the analysis and design of the components made of FGMs.

In last thirty years, the interest in the study of an FGM hollow cylinder under thermal and/or mechanical loads has been growing rapidly. Ootao et al. (1989, 1991, 1995) analyzed the plane transient thermal stress problem of the laminated hollow cylinder under the symmetrical and asymmetrical heating loads by using the Laplace transform, and extended to the FGM hollow cylinder. Subsequently, they discussed the corresponding three-dimensional axisymmetric thermo-elastic problem by applying the Fourier cosine transform and the Laplace transform to the temperature field and the thermo-elastic potential function, respectively. Reddy and Chin (1998) and Praveen et al. (1999) investigated the dynamic thermo-elastic response of functionally graded cylinders and developed a finite element model in which the coupling effect on the temperature and stress fields was included. In their studies the material properties were expressed as a simple power law distribution and varied with temperature. Considering an FGM hollow cylinder whose Young's modulus varied in the radial direction according to a power law relation, the close-form solutions for the stresses and displacements in the cylinder subjected to uniform pressures on the inner and/or outer surfaces were given by Horgan and Chan (1999) and by Tutuncu and Ozturk (2001), respectively. Kim and Noda (2002) studied the two-dimensional unsteady thermal stresses in an infinite FGM hollow cylinder based on the laminate theory and the Green function approach. Solving the Navier equation directly, Jabbari et al. (2002) derived explicitly the analytical solutions of the one-dimensional steady thermal stress in a thick FGM hollow cylinder whose material properties, except Poisson's ratio, were expressed as the power functions of *r*. Using the displacement method, Xiang et al. (2006) considered an FGM hollow cylinder with constant Poisson's ratio and linearly or exponentially varying Young's modulus and provided the exact solutions for the cylinder subjected to uniform pressures on the inner and outer surfaces by solving hyper-geometric equations. Dryden and Jayaraman (2006) analyzed a nonhomogeneous pipe under the uniform pressure and obtained the explicit expressions of the stresses in terms of the Kummer functions. In their study, a more general expression to define the variation of Young's modulus along the radial direction was proposed. Haughton

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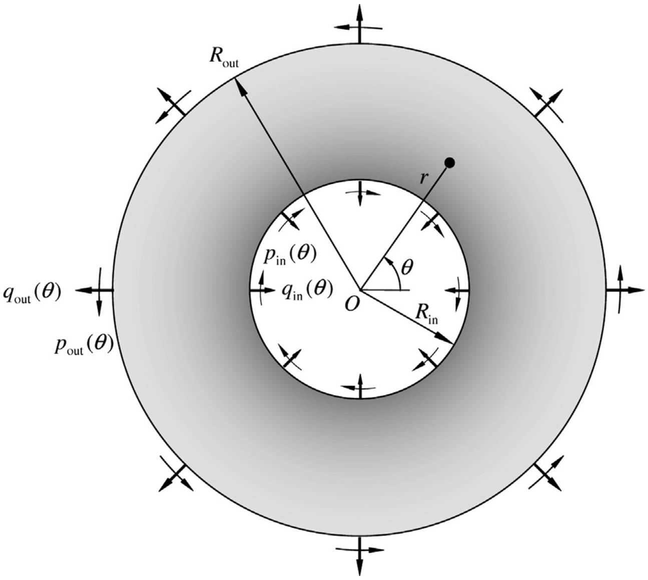

Fig. 1. Schematic diagram of the FGM hollow cylinder.

and Merodio (2009) studied the bifurcation of inflated thin-walled cylinders under axial loading, in which the localized strain softening had been identified as associated with Marfan's syndrome. More recently, Khoshgoftar et al. (2013) gave the exact solution of an FGM thick cylinder with finite length under non-uniform pressure along the length of the cylinder by using the first-order shear deformation theory.

Most of the above-mentioned studies are focused on the axisymmetric problem. There is a little literature concerning the case of the non-uniform load which is encountered in many practical problems. In this work, we consider an FGM circular hollow cylinder subjected to arbitrarily nonaxisymmetric loads on the inner and outer surfaces. The material properties of the cylinder may vary arbitrarily along the thickness. Based on the method of piece-wise homogeneous layers, the analytical solutions for the stresses and displacements are derived via the complex function theory of Muskhelishvili (1977).

2. Model and basic equations

Consider an infinitely long hollow cylinder with uniform cross-section, R_{in} and R_{out} being the inner and outer radii of the cylinder, respectively, as shown in Fig. 1. The origin O of the polar coordinate system is placed at the center of the cross-section, and an arbitrary point inside the cylinder is denoted by (r, θ). The cylinder is assumed to be made of a linearly elastic, isotropic material with the material properties varying arbitrarily through the radial direction, and submitted to arbitrarily nonaxisymmetric loads $q_{\mathrm{in}}(\theta)$, $p_{\mathrm{in}}(\theta)$ and $q_{\mathrm{out}}(\theta)$, $p_{\mathrm{out}}(\theta)$ on the inner and outer surfaces, respectively.

For plane strain problem, the stresses (σ_r, σ_θ, and $\tau_{r\theta}$), the displacements (u_r and u_θ), and the surface forces per unit area (f_r and f_θ) on a curve in polar coordinates (r, θ) can be expressed in terms of two complex potentials $\phi(z)$ and $\psi(z)$ (Muskhelishvili, 1977) as

$$\sigma_r + \sigma_\theta = 2[\phi'(z) + \overline{\phi'(z)}],\ \sigma_\theta - \sigma_r + 2\mathrm{i}\tau_{r\theta} = 2[\bar{z}\phi''(z) + \psi'(z)]\mathrm{e}^{2\mathrm{i}\theta}, \tag{1}$$

$$2\mu(u_r + \mathrm{i}u_\theta) = [\kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}]\mathrm{e}^{-\mathrm{i}\theta}, \tag{2}$$

$$\mathrm{i}\int_L (f_r + \mathrm{i}f_\theta)\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}s = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}, \tag{3}$$

where $z = r\mathrm{e}^{\mathrm{i}\theta}$ is the complex coordinate variable, μ is the shear modulus, $\kappa = 3 - 4\nu$ with ν being Poisson's ratio, the bar represents a complex conjugate, and the prime denotes differentiation with respect to the argument.

For the problem as shown in Fig. 1, the boundary conditions on the inner and outer surfaces are

$$\sigma_r + \mathrm{i}\tau_{r\theta} = q_{\mathrm{in}}(\theta) + \mathrm{i}p_{\mathrm{in}}(\theta) \quad \text{when} \quad r = R_{\mathrm{in}}, \tag{4a}$$

$$\sigma_r + \mathrm{i}\tau_{r\theta} = q_{\mathrm{out}}(\theta) + \mathrm{i}p_{\mathrm{out}}(\theta) \quad \text{when} \quad r = R_{\mathrm{out}}. \tag{4b}$$

3. Analytical solutions

Assume that the loads can be expressed as

$$q_{\mathrm{in}}(\theta) = q_0^{\mathrm{in}} + \sum_{n=1}^{\infty}(q_n^{\mathrm{in}} \cos n\theta + \tilde{q}_n^{\mathrm{in}} \sin n\theta), \tag{5a}$$