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Direct way of computing the variability of modal assurance criteria

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ABSTRACT

The study considers the variability of modal assurance criteria (MAC) for undamped systems with distinct eigenvalues, which is often approximated by the first-order Taylor series. So far all the methods for computing the sensitivity of MAC value belong to the indirect method. This study presents a direct method carried out by constructing a Lagrange function. When the number of design variables is larger than one, the proposed method will be efficient in computational time and storage capacity. The validity is illustrated using a numerical example.

1. Introduction

Eigenproblem is of fundamental importance for structural dynamics due to the fact that the eigensolutions (natural frequencies and mode shapes) represent the dynamic characteristics of structures. During the design process of structural dynamical systems, it is usually required to make changes in the design parameters so that the design is optimal. Variations in the structural design variables lead to changes in these eigensolutions and corresponding responses. These variations (e.g. first-order Taylor series approximations) can be used as quantitative measures that provide the means for assessing the state of the damage or the health of engineering structure, dynamic model updating and many other applications. It has been pointed out (Mottershead et al., 2011) that the sensitivity based method is probably the most successful of the many approaches to the problem of finite element (FE) model updating and has developed into a mature technology applied successfully for the correction of large-scale engineering models. For engineering application, modal data (natural frequencies and mode shapes) extracted from measured frequency response data have found broad application as a target for model updating. Under such circumstances, it should ensure that the analytical and measured eigenvalues correspond to the same mode shape. This process, often referred to a mode pairing, may be achieved by carrying out a correlation utilizing modal assurance criteria (MAC) (Allemang, 2003; Friswell and Mottershead, 1995). The MAC, originated from the need for a quality assurance indicator for experimental mode shapes that are estimated from measured frequency response functions (FRFs), has become an integral part of many engineering design methodologies including model updating (Bohle and Fritzen, 2003; Modak et al., 2000; Mottershead et al., 2011; Ribeiro et al., 2012), structural optimization (Ouisse and Cogan, 2010; Zhang et al., 2012), structural health monitoring (Koh and Dyke, 2007), dynamic modification (Massa et al., 2011) and many other applications. With the recent developments of model updating (some existing commercial codes are available, e.g., LMS Virtual.Lab, FEMtools), the need to consider the computational efficiency and accuracy of numerical methods for parametric variability analysis of MAC value is more than ever before. For dynamic model updating, a sequential quadratic programming algorithm is often used and this target function $J(\Delta\mathbf{p})$ which has to be minimized is usually defined as (Bohle and Fritzen, 2003)

$$J(\Delta\mathbf{p}) = \sum_{i=1}^{n} w_{MAC,i}[1 - \mathrm{MAC}_{ii}(\Delta\mathbf{p})]^2 + \sum_{i=1}^{n} w_{freq,i}\left(\frac{\omega_i(\Delta\mathbf{p}) - \omega_i^e}{\omega_i^e}\right)^2 \tag{1}$$

Here $\Delta\mathbf{p}$ indicates the variation of design variable vector $\mathbf{p}$, $w_{MAC,i}$ and $w_{freq,i}$ are weighting factors. The numerical value of n is the number of active modes. MAC_{ii} is the ith diagonal element of MAC matrix. ω_i and ω_i^e denote the analytical and measured natural circular frequencies, respectively. Suppose the number of design variables is q, we denote the design variable vector $\mathbf{p}$ as

$$\mathbf{p} = \{p_1, p_2, \ldots, p_q\}$$

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where p_k for $k = 1, 2, \ldots, q$, are the design variables. These variabilities are often approximated by first-order Taylor series $\omega_i(\mathbf{p}_0 + \Delta\mathbf{p}) = \omega_i(\mathbf{p}_0) + \frac{d\omega_i(\mathbf{p}_0)}{d\mathbf{p}}(\Delta\mathbf{p})^T$, $\mathrm{MAC}_{ii}(\mathbf{p}_0 + \Delta\mathbf{p}) = \mathrm{MAC}_{ii}(\mathbf{p}_0) + \frac{d\mathrm{MAC}_{ii}(\mathbf{p}_0)}{d\mathbf{p}}(\Delta\mathbf{p})^T$, where $\mathbf{p}_0$ denotes the initial value of design variable vector $\mathbf{p}$. As can be seen, these approximations need the information of sensitivity analysis. Computational methods for design sensitivity analysis have received much attention over the past four decades, particularly those related to the eigenproblems. Although computing eigenvalue sensitivity is straightforward, the derivatives of MAC values raise several challenges, due to the fact that they involve the derivatives of mode shapes.

In general, the choice between the different options for the method of design sensitivity analysis of MAC value by four criteria: (a) the accuracy of the computations; (b) the computational time taken; (c) the storage capacity involved; and (d) the ease of implementation. For the moment, the computation of the sensitivity of MAC value can be only calculated in an indirect way by substituting the sensitivity of mode shape into the formula of it. In a word, all the methods belong to the indirect method.

The indirect method is based on the existing sensitivity method of mode shape. Design sensitivity analysis of eigenproblem has received much attention over the past four decades. Several methods have been developed for the calculation of the sensitivity of mode shape including the model method (Adhikari, 1999, 2002; Adhikari and Friswell, 2001; Fox and Kapoor, 1968), the finite difference method (Vanhonacker, 1980), Nelson's method (Adhikari and Friswell, 2006; Friswell and Adhikari, 2000; Li et al., 2012a, 2013c,d; Nelson, 1976), the modified modal method (Li et al., 2013a; Moon et al., 2004; Wang, 1991), the algebraic method (Chouchane et al., 2007; Guedria et al., 2006; Lee and Jung, 1997; Lee et al., 1999; Li et al., 2012b).

This study is aimed to efficiently calculate the derivatives of MAC values by a direct way. We present a direct method to calculate the sensitivity of MAC value for undamped systems with distinct eigenvalues. This method is presented by constructing a Lagrange function. Once the Lagrange multipliers are evaluated, the sensitivity of MAC value can be computed directly, regardless of the number of design variables. Therefore the proposed method will be efficient in computational time and storage capacity for the number of design variables is larger than one.

2. Theoretical background

The equations of motion for the free vibration of a linear undamped discrete system with N degrees of freedom (DOF) can be given by

$$\mathbf{K}\boldsymbol{\phi}_i = \lambda_i \mathbf{M}\boldsymbol{\phi}_i \tag{2}$$

where $\mathbf{M}$ and $\mathbf{K} \in \mathbb{R}^{N\times N}$ are the mass and stiffness matrices, respectively, whose components depend continuously on the design variable vector $\mathbf{p}$ (the design variables may be material properties, geometrical properties, boundary conditions, lumped masses ant etc.) and $\boldsymbol{\Phi}_i$ is the ith mode shape. In this limit case, the eigenvalues $\lambda_i = \omega_i^2$. Since damping is not considered in this study, the discussed methods in the following text are restricted to lightly damped structures wherein the undamped real modal model can effectively represent the damped structure.

This paper considers the problem for undamped symmetric systems with distinct eigenvalues. As the mass matrix $\mathbf{M}$ is nonsingular, the mode shapes are usually normalized as:

$$\boldsymbol{\Phi}_i^T \mathbf{M}\boldsymbol{\phi}_i = 1 \tag{3}$$

MAC value is widely used as a tool of vector correlation and given by

$$\mathrm{MAC}_{ij} = \frac{(\boldsymbol{\psi}_j^T \boldsymbol{\phi}_i)^2}{(\boldsymbol{\psi}_j^T \boldsymbol{\psi}_j)(\boldsymbol{\phi}_i^T \boldsymbol{\phi}_i)} \tag{4}$$

where $\boldsymbol{\psi}_j$ denotes the jth measured mode shape. The MAC value close to unity indicates similarity whereas MAC value close to zero indicates no similarity.

The derivatives of MAC values with respect to the design variable vector $\mathbf{p}$ can be obtained from Eq. (4) and given by

$$\frac{d\mathrm{MAC}_{ij}}{d\mathbf{p}} = \frac{2(\boldsymbol{\psi}_j^T \boldsymbol{\phi}_i)\boldsymbol{\psi}_j^T}{(\boldsymbol{\psi}_j^T \boldsymbol{\psi}_j)(\boldsymbol{\phi}_i^T \boldsymbol{\phi}_i)} \frac{d\boldsymbol{\phi}_i}{d\mathbf{p}} - \frac{2(\boldsymbol{\psi}_j^T \boldsymbol{\phi}_i)^2 \boldsymbol{\phi}_i^T}{(\boldsymbol{\psi}_j^T \boldsymbol{\psi}_j)(\boldsymbol{\phi}_i^T \boldsymbol{\phi}_i)^2} \frac{d\boldsymbol{\phi}_i}{d\mathbf{p}} \tag{5}$$

Several methods have been developed for the calculation of the sensitivity of mode shape. Hence the derivatives of MAC value can be calculated in an indirect way by substituting the sensitivity of mode shape into the above formula. However, the algorithm is lengthy and needs heavy computational cost.

3. Structural sensitivity analysis of eigenvalue problems

The derivatives of eigenvalues and undamped circular frequencies can be easily obtained by differentiation of the undamped eigenproblem (2)

$$\frac{\partial\lambda_i}{\partial p_k} = \boldsymbol{\Phi}_i^T \left(\frac{\partial\mathbf{K}}{\partial p_k} - \lambda_i \frac{\partial\mathbf{M}}{\partial p_k} \right) \boldsymbol{\phi}_i$$

$$\text{and} \quad \frac{\partial\omega_i}{\partial p_k} = \frac{1}{2\omega_i}\frac{\partial\lambda_i}{\partial p_k} \quad \text{for} \quad k = 1, 2, \ldots, q \tag{6}$$

The derivatives of mode shapes cannot be found directly due to the fact that it needs to overcome the singular problem. Many methods have been developed to compute the sensitivities of mode shapes.

3.1. The modal method

The firstly derived expressions of the first-order derivatives of modes can be found in (Fox and Kapoor, 1968) by expressing these derivatives as a linear combination of all the modes

$$\frac{\partial\boldsymbol{\Phi}_i}{\partial p_k} = \sum_{j=1}^{N} c_j \boldsymbol{\Phi}_j \quad \text{for} \quad k = 1, 2, \ldots, q \tag{7}$$

in which

$$c_j = \begin{cases} \dfrac{\boldsymbol{\Phi}_i^T}{\lambda_j - \lambda_i}\left(-\dfrac{\partial\mathbf{K}}{\partial p_k} + \lambda_i \dfrac{\partial\mathbf{M}}{\partial p_k} \right) \boldsymbol{\phi}_i & \text{if} j \neq i \\ -0.5\boldsymbol{\Phi}_i^T \dfrac{\partial\mathbf{M}}{\partial p_k} \boldsymbol{\phi}_j & \text{if} j = i \end{cases} \tag{8}$$

The modal method is widely applied in engineering due to its simplicity of implementation. However, this method needs all the mode shapes to obtain an exact sensitivity of each mode shape.

3.2. Nelson's method

A technique has been developed (Nelson, 1976) for the evaluation of the derivative of each mode which only requires the mode of interest. The mode sensitivity is expressed as a particular solution $\mathbf{V}_i$ and a homogeneous solution $\boldsymbol{\phi}_i d_i$ of the singularity problem(9) $\frac{\partial\boldsymbol{\phi}_i}{\partial p_k} = \mathbf{V}_i + \boldsymbol{\phi}_i d_i \quad \text{for} \quad k = 1, 2, \ldots, q$ where d_i is an coefficient needed to be determined. The particular solution can