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Evaluation of elastic compensation using elastic/plastic rotating circular disk problems

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ABSTRACT

A version of elastic compensation is evaluated in the context of stress and deformation analysis of elastic/plastic rotating circular disks of both constant and variable thicknesses undergoing small deflections. An iterative incremental method is combined with finite difference methodology to generate information about the entire quasistatic loading histories of such disks. The evaluation process involves comparison of representative numerical results with corresponding predictions existing in the literature.

1. Introduction

The method of elastic compensation has been proposed as a way to minimize simulation times by using linear elasticity modules in commercial finite element packages to perform elastic/plastic stress and deformation analysis approximately. As can be seen from the papers of Seshadri (1991), Mohamed et al. (1999), Desikan and Sethuraman (2000), Reinhardt and Mangalaramanan (2001), and Yang et al. (2012); there are a number of variants of this methodology which have been given different names. Here elastic compensation is used to refer to all such procedures. The basic idea is to iteratively modify the elastic properties appearing in a linear elastic formulation to simulate yielding. A natural way to do this is to use a specific type of deformation plasticity model. In the present paper the evaluation of this approach to elastic compensation, as exemplified by the papers by Khalili et al. (2004) on trusses and by Wu et al. (2003) and Upadrasta et al. (2006) on plates, is continued using test problems associated with rotating discs. It should be emphasized that the purpose of the present work is neither to report new information about rotating disks nor to develop the elastic compensation approach further, but to contribute to the evaluation of elastic compensation. Gaining confidence in an approximate method requires testing in a variety of situations. This is especially true in the case of elastic compensation, where several different approaches have been proposed.

Classical work on the elastic/plastic analysis of stresses and deformations in rotating discs was summarized in early plasticity texts such as Hoffman and Sachs (1953). Representative of more recent contributions are the papers by Güven (1997, 1998), Rees (1999), Eraslan (2002a,b), Eraslan and Orcan (2002), Eraslan and Argeso (2002), Eraslan et al. (2005), Vivio and Vullo (2010), and Aleksandrova (2012). This extensive literature is helpful in the evaluation of elastic compensation and some of the results reported in these papers will be used subsequently for that purpose.

The reminder of the paper is organized as follows: Section 2 presents a set of governing equations for elastic compensation based stress and deformation analysis of rotating discs using the usual plane stress approach. Section 3 presents and discusses a representative sample of numerical results. Section 4 presents a summary of the work and recapitulates the most important conclusions.

2. Formulation

Consider a linearly elastic annular disk of inner radius r_i, outer radius r_o, thickness h, modulus of elasticity E, Poisson's ratio ν, and mass density ρ rotating with constant angular velocity ω in a horizontal plane. It will be convenient to describe the behavior of the disk using the cylindrical coordinates r, θ, and z with the latter being the axis of rotation and the r, θ plane being the middle surface of the disk. The respective radial and circumferential normal stresses will be denoted by σ_r and σ_θ. It will be assumed in all subsequent work

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that all quantities depend only on r and that deformations are small. Differentiation with respect to r will be denoted by a superposed prime.

A radial balance of linear momentum leads to

$$(hr\sigma_r)' - h\sigma_\theta + h\rho\omega^2 r^2 = 0 \tag{1}$$

The plane stress forms of Hooke's law for an isotropic linearly elastic material are

$$\sigma_r = \frac{E(u' + \nu u/r)}{1-\nu^2}, \quad \sigma_\theta = \frac{E(u/r + \nu u')}{1-\nu^2} \tag{2}$$

where u is the radial displacement. Substituting Eq. (2) into Eq. (1) and regarding h, E, ν, and ρ as functions of r creates the differential equation

$$u'' + \left(\frac{1}{r} + \frac{h'}{h} + \frac{E'}{E} + \frac{2\nu\nu'}{1-\nu^2}\right)u' + \left(-\frac{1}{r} + \nu\left(\frac{h'}{h} + \frac{E'}{E}\right) + \frac{\nu'(1+\nu^2)}{1-\nu^2}\frac{u}{r}\right) + \frac{(1-\nu^2)\rho\omega^2 r}{E} = 0 \tag{3}$$

Elastic compensation (iteratively modifying the elastic properties used in a linearly elastic formulation to simulate yielding) is accomplished herein by treating ν and ρ as constants and using a generalization of a one-dimensional relationship developed in Goldberg and Richard (1963) and Richard and Abbott (1975) for the effective modulus of elasticity, namely

$$E = E_0\left(\frac{1-S}{(1+I^M)^{1/M}} + S\right) \tag{4}$$

where

$$I = \frac{E_0(1-S)\varepsilon}{\sigma_I}; \tag{5}$$

E_0, S, M, and σ_I are constants; and ε is the strain invariant

$$\varepsilon = \left(\frac{u'^2 + (u/r)^2 + 2\nu u u'/r}{1-\nu^2}\right)^{1/2} \tag{6}$$

Combining Eqs. (3)–(6) creates a nonlinear differential equation which, in general, requires numerical solution. Eqs. (4)–(6) can be thought of as describing a strain invariant based deformation model of plasticity. Their use automatically modifies the elastic properties of the material in such a way as to simulate elastic/plastic response. Thus, as indicated earlier, this is a natural way to implement elastic compensation. As with all deformation plasticity models, this model can be applied with confidence only in monotonic loading situations. Deformation plasticity models of this kind are not new (see, for instance, Naghdi (1952) and Novozhilov (1953)), but do not appear to have been widely adopted. A recent application is discussed by Goel et al. (2011) who use a model similar to but more sophisticated than that described by Eqs. (4)–(6). The use of nonlinear elasticity modules in commercial finite element packages to mimic elastic/plastic response can also sometimes be interpreted as elastic compensation of the type discussed herein.

The uniaxial version of the stress/strain relation discussed above (see Fig. 3 of Wu et al., 2003; with n and R therein playing the respective roles of M and S in the present work) exhibits a linear elastic asymptote with modulus of elasticity E_0 and a linear strain hardening asymptote with effective yield stress σ_I and plastic modulus SE_0 connected by an elbow controlled by the value of M (the larger the M, the smaller the elbow). Varying M makes it possible to model materials both with and without definite yield points (stainless steel being an example of the latter). The two asymptotes intersect at $I = 1$. For large values of M (small elbows), therefore, $I = 1$ is good indicator of yielding and will be used as such subsequently. In particular, the corresponding stress $\sigma_I/(1-S)$ will be identified with the yield stresses used by various authors for the purpose of making the quantitative comparisons reported below. The definition of yielding is somewhat arbitrary because Eqs. (4) and (5) predict a gradual (rather than abrupt) transition from elastic to plastic response.

The boundary conditions employed in the present work were

$$u(r_i) = 0 \quad \text{or} \quad \sigma_r(r_i) = 0 \quad \text{and} \quad \sigma_r(r_o) = 0 \tag{7}$$

The combination of the first and third of Eq. (7) (fixed inner edge, free outer edge) will be referred to as BCI hereafter while the combination of the second and third of Eq. (7) (free inner edge, free outer edge) will be referred to as BCII. A solid disk with a free outer edge can, of course, be represented by BCI with $r_i = 0$. While the formulation given above is correct for any thickness profile, all results to be presented subsequently are based on the three parameter thickness profile

$$h = h_0\left(\frac{1-nr}{r_i}\right)^k \tag{8}$$

For convenience in subsequent numerical work it is helpful to define the dimensionless quantities

$$R = \frac{r}{r_o}, \quad R_i = \frac{r_i}{r_o}, \quad H = \frac{h}{r_o}, \quad Y = \frac{E}{E_0} \tag{9}$$

$$U = \frac{E_0 u}{\rho\omega^2 r_o^3}, \quad \Sigma_r = \frac{\sigma_r}{\rho\omega^2 r_o^2}, \quad \Sigma_\theta = \frac{\sigma_\theta}{\rho\omega^2 r_o^2} \tag{10}$$

It can then be shown that U, Σ_r, and Σ_θ depend on only the dimensionless angular velocity

$$\Omega = \omega r_o\left(\frac{\rho}{\sigma_I}\right)^{1/2}, \tag{11}$$

the dimensionless numbers M and S appearing in Eq. (4), and the dimensionless numbers k and n appearing in Eq. (8).

Elastic compensation was accomplished iteratively using the dimensionless forms of Eqs. (3)–(7). At a given iteration the differential equation was linearized about the result from the previous iteration (previous Ω at the first iteration for a new Ω) and the resulting linear variable coefficient differential equation was solved numerically, with iteration continuing until convergence was achieved to sufficient accuracy. Then Ω was incremented and the iterative process restarted. The calculation began with a small value of Ω (to insure elastic behavior) and proceeded successively to larger Ω values. The numerical solution of the linear variable coefficient differential equation was performed by dividing the radial coordinate into $N-1$ segments of equal length by N grid points, representing the first and second derivatives appearing in Eq. (3) by appropriate three point central difference quotients and the first derivatives appearing in the second and third of Eq. (7) by respective three point forward and backward difference quotients, and solving the resulting set of N linear tri-diagonal algebraic equations by the Thomas algorithm (Potter's method). Verification of the numerical method was carried out by comparing its predictions with closed form solutions for both constant and variable thickness elastic discs reported by Timoshenko and Goodier (1951), numerical solutions for variable thickness elastic disks presented in Fig. 9a and c of Eraslan and Argeso (2002), and numerical solutions for variable thickness elastic disks presented in Fig. 2a and c of Eraslan and Orcan (2002). Excellent agreement was observed in all cases. Several values of N were employed in the simulations and it was found that grid independence was achieved for $N = 1001$ in all cases.