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Modified Mindlin plate theory and shear locking-free finite element formulation

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ABSTRACT

The basic equations of the Mindlin theory are specified as starting point for its modification in which total deflection and rotations are split into pure bending deflection and shear deflection with bending angles of rotation, and in-plane shear angles. The equilibrium equations of the former displacement field are split into one partial differential equation for flexural vibrations. In the latter case two differential equations for in-plane shear vibrations are obtained, which are similar to the well-known membrane equations. Rectangular shear locking-free finite element for flexural vibrations is developed. For in-plane shear vibrations ordinary membrane finite elements can be used. Application of the modified Mindlin theory is illustrated in a case of simply supported square plate. Problems are solved analytically and by FEM and the obtained results are compared with the relevant ones available in the literature.

1. Introduction

The first works on thick plate theory are those of Reissner and Mindlin from 1945 and 1951, (Reissner, 1945) and (Mindlin, 1951), respectively. This challenging problem has been a subject of investigation by many researchers, both mathematicians and engineers. Very large number of concepts has been worked out during that long period (Liew et al., 1995). Analytical and numerical methods have been applied. When the finite element method (FEM) came to use, it was also applied for thick plate static and dynamic analysis (Hughes, 1987).

In the Mindlin theory shear deformations are taken into account, and application of ordinary low-order finite element is not capable to reproduce the pure bending modes in the limit case of thin plate. This shear locking problem arises due to inadequate dependence among transverse deflection and two rotations. In order to overcome this problem, quite large number of procedures has been developed in recent years. Most of them utilize a mixed formulation, by linking plate deflection field to the angles of rotations (Lee and Wong, 1982; Auricchio and Taylor, 1995; Lovadina, 1998). These formulations are rather complex and time consuming. Another method is the Assumed Natural Strain (ANS) in which shear strains at discrete collocation points are determined from the displacements and interpolated over the element surface with specific shape functions (Hughes and Tezduyar, 1981; Bathe, 1996; Zienkiewicz and Taylor, 2000). The Discrete Shear Gap method (DSG) is similar to the ANS since the course of certain strains is modified within the finite element (Bletzinger et al., 2000). The lack of collocation points makes application of DSG independent of the order and form of the finite elements as the main difference from the ANS. The DSG method has been recently used in combination with Edge–based Smoothed FE Method (ES-FEM) (Nguyen-Xuan et al., 2010), as a particular meshless method (Liu et al., 2009).

Motivated by the above state-of-the art an investigation of the problem has been undertaken. The well-known Mindlin theory is modified in such a way that independent total deflection and angles of rotations are split into pure bending deflection and shear deflection, and bending rotations and in-plane shear angles, respectively, based on the idea elaborated in (Senjanović et al., 2013). Since shear deflection and bending rotations depend on bending deflection, Mindlin mathematical model with 3 DOFs is decomposed into single DOF bending model and double DOF shear model. Following the modified Mindlin theory shear locking-free finite element formulation is given. Three numerical examples for thick plate with different boundary conditions are analyzed and the results are compared with those from relevant literature.

2. Basic equations of Mindlin plate theory

The Mindlin theory deals with plate deflection, w, and angles of cross-section rotation ψ_x and ψ_y. The following relations between bending moments, torsional moments and transverse shear forces,

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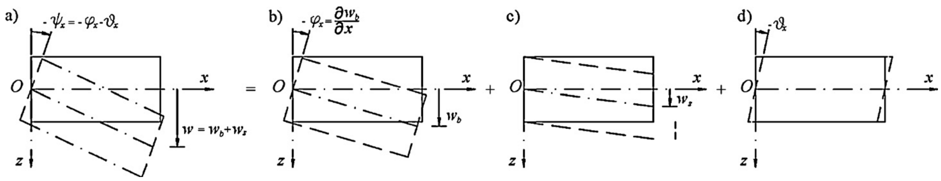

Fig. 1. Thick plate displacements; (a) total deflection and rotation w, ψ_x; (b) pure bending deflection and rotation w_b, φ_x; (c) transverse shear deflection w_s and (d) in-plane shear angle ϑ_x.

and displacements are specified

$$M_x = D\left(\frac{\partial \psi_x}{\partial x} + \nu\frac{\partial \psi_y}{\partial y}\right), \quad M_y = D\left(\frac{\partial \psi_y}{\partial y} + \nu\frac{\partial \psi_x}{\partial x}\right),$$
$$M_{xy} = M_{yx} = \frac{1}{2}(1-\nu)D\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right),$$
$$Q_x = S\left(\frac{\partial w}{\partial x} + \psi_x\right), \quad Q_y = S\left(\frac{\partial w}{\partial y} + \psi_y\right), \tag{1}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad S = kGh, \tag{2}$$

is plate flexural rigidity and shear rigidity, respectively, h is plate thickness, k is shear coefficient, E and $G = E/(2(1+\nu))$ are Young's and shear modulus, respectively, while ν is Poisson's ratio. The plate is loaded with transverse inertia load and distributed inertia moments

$$q = -\bar{m}\frac{\partial^2 w}{\partial t^2}, \quad m_x = J\frac{\partial^2 \psi_x}{\partial t^2}, \quad m_y = J\frac{\partial^2 \psi_y}{\partial t^2}, \tag{3}$$

where $\bar{m} = \rho h$ and $J = \rho h^3/12$ is plate specific mass per unit area and its moment of inertia, respectively and ρ is mass density. Equilibrium of moments about y and x axis and transverse forces leads to

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = m_x, \quad \frac{\partial M_y}{\partial y} + \frac{\partial M_{yx}}{\partial x} - Q_y = m_y,$$
$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q. \tag{4}$$

By substituting Eqs. (1) and (3) into (4) one arrives at three differential equations of motion

$$\frac{D}{S}\left[\frac{\partial^2 \psi_x}{\partial x^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 \psi_x}{\partial y^2} + \frac{1}{2}(1+\nu)\frac{\partial^2 \psi_y}{\partial x \partial y}\right] - \left(\frac{\partial w}{\partial x} + \psi_x\right) - \frac{J}{S}\frac{\partial^2 \psi_x}{\partial t^2} = 0, \quad \frac{D}{S}\left[\frac{\partial^2 \psi_y}{\partial y^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 \psi_y}{\partial x^2} + \frac{1}{2}(1+\nu)\frac{\partial^2 \psi_x}{\partial x \partial y}\right]$$
$$- \left(\frac{\partial w}{\partial y} + \psi_y\right) - \frac{J}{S}\frac{\partial^2 \psi_y}{\partial t^2} = 0, \quad \Delta w + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} - \frac{\bar{m}}{S}\frac{\partial^2 w}{\partial t^2} = 0, \tag{5}$$

where $\Delta(\cdot)$ is the Laplace differential operator. Eq. (5) generally represent starting point for further development of Mindlin theory and its variants.

3. Modification of Mindlin theory

The main idea is to split general displacements w, ψ_x and ψ_y, Fig. 1a, into their constitutive parts, as shown in Fig. 1b–d. Total deflection consists of bending deflection and contribution of transverse shear, while angles of plate cross-section slope are a result of angles of rotation, Fig. 1b, due to pure bending and shear angles

$$w = w_b + w_s, \quad \psi_x = -\frac{\partial w_b}{\partial x} + \vartheta_x, \quad \psi_y = -\frac{\partial w_b}{\partial y} + \vartheta_y. \tag{6}$$

By introducing (6) into Eq. (5) it is possible to separate variables of two different displacement fields

$$\frac{\partial}{\partial x}\left(\frac{D}{S}\Delta w_b - \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2} + w_s\right) = \frac{D}{S}\left[\frac{\partial^2 \vartheta_x}{\partial x^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 \vartheta_x}{\partial y^2} + \frac{1}{2}(1+\nu)\frac{\partial^2 \vartheta_y}{\partial x \partial y}\right] - \vartheta_x - \frac{J}{S}\frac{\partial^2 \vartheta_x}{\partial t^2}, \tag{7}$$

$$\frac{\partial}{\partial y}\left(\frac{D}{S}\Delta w_b - \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2} + w_s\right) = \frac{D}{S}\left[\frac{\partial^2 \vartheta_y}{\partial y^2} + \frac{1}{2}(1-\nu)\frac{\partial^2 \vartheta_y}{\partial x^2} + \frac{1}{2}(1+\nu)\frac{\partial^2 \vartheta_x}{\partial x \partial y}\right] - \vartheta_y - \frac{J}{S}\frac{\partial^2 \vartheta_y}{\partial t^2}, \tag{8}$$

$$\Delta w_s - \frac{\bar{m}}{S}\frac{\partial^2}{\partial t^2}(w_b + w_s) = -\left(\frac{\partial \vartheta_x}{\partial x} + \frac{\partial \vartheta_y}{\partial y}\right). \tag{9}$$

Eqs. (7) and (8) can be presented in the form

$$\frac{\partial F(w_b, w_s)}{\partial x} = g_1(\vartheta_x, \vartheta_y), \quad \frac{\partial F(w_b, w_s)}{\partial y} = g_2(\vartheta_x, \vartheta_y) \tag{10}$$

and their integrals per x and y read $F = \int g_1 dx + f(y, t) = G_1$ and $F = \int g_2 dy + f(x, t) = G_2$, respectively. That implies identity of functions G_1 and G_2, which is not possible due to structure of g_1 and g_2 in (7) and (8). The reasonable solution is that both functions g_1 and g_2 are set to zero. Consequently $\partial F/\partial x$ and $\partial F/\partial y$ are also zero and their integrals $F = f(y, t)$ and $F = f(x, t)$ have to be the same, i.e. $f(y, t) = f(x, t) = f(t)$. Since $f(t)$ represents rigid body motion, it can be ignored in vibration analysis.

As a result of the above consideration, the following relation from Eqs. (7) and (8) yields

$$w_s = -\frac{D}{S}\Delta w_b + \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}. \tag{11}$$

Furthermore, by substituting Eq. (11) into (9) one arrives at differential equation for flexural vibrations

$$\Delta\Delta w_b - \frac{J}{D}\left(1 + \frac{\bar{m}D}{JS}\right)\frac{\partial^2}{\partial t^2}\Delta w_b + \frac{\bar{m}}{D}\frac{\partial^2}{\partial t^2}\left(w_b + \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}\right) = \frac{S}{D}\left(\frac{\partial \vartheta_x}{\partial x} + \frac{\partial \vartheta_y}{\partial y}\right). \tag{12}$$

Once w_b is determined, the total deflection reads, according to Eqs. (6) and (11),

$$w = w_b - \frac{D}{S}\Delta w_b + \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}. \tag{13}$$