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An Euler–Bernoulli-like finite element method for Timoshenko beams

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ABSTRACT

In this paper a new finite element approach for the solution of the Timoshenko beam is shown. Similarly to the Euler–Bernoulli beam theory, it has been considered a single fourth order differential equation governs the equilibrium of the Timoshenko beam. The results obtained by this approach are very good, both in terms of accuracy and computational effort.

1. Introduction

In structural mechanics, the Euler–Bernoulli beam model (EBBM) represents the most widely used theory for modelling the behaviour of the beam. This theory was extended by Timoshenko (1922) in order to account for the deformation shear effects, that are neglected in the EBBM. The corresponding extended theory is usually known as Timoshenko beam model (TBM). This extension gave small correction for slender beams and significant differences in case of short beams.

A large number of Finite Element (FE) methods applied to the TBM appeared in the literature in the last fifty years (Heiliger and Reddy, 1988; Nickell and Secor, 1972; Prathap and Bhashyam, 1982; Reddy, 1993; Tessler and Dong, 1981). Most of them differ from one other in the choice of interpolation functions used for the deflection, w, and rotation, φ, or in the weak form used for the formulation of the FE model. Some of them are based on the equal interpolation order for w and φ. But these last ones lead to the so-called shear locking problem (Prathap and Bhashyam, 1982; Reddy, 1993), due to the inconsistency of the same order of interpolation used for quantities having different dimensions. This problem was partially solved by some approaches, as well described in Reddy (1993). The most known are the Consistent Interpolation Element (CIE) method and the Reduced Integration Element (RIE) method; a particular mention must be given to the Interdependent Interpolation Element (IIE) method (Reddy, 1997), as well.

It is important to note that most authors consider the problem as governed by two equilibrium equations expressed in terms of deflection and rotation, whereas only few of them evidence the possibility of coupling these equations in terms of deflection, w (Craig, 1981; Antes, 2003; Li, 2008). Moreover, it is possible to obtain a single governing differential equation expressed in terms of rotation, φ, too. These considerations are fundamental for the present work. The application of the weak formulation to the single differential equation governing the TBM allows us to obtain a FE approach which is an evident extension of the FE approach applied to the EBBM. As a matter of fact, for example, the primary variable to be interpolated is only one and the interpolation is made through Hermite polynomials, as in the EBBM.

2. Preliminary concepts

The equations governing the elastic behaviour of a Timoshenko beam are:

(a) The equilibrium ones:

$$T'(x) = -q(x); \quad M'(x) = T(x) \tag{1a,b}$$

where $q(x)$ is the transversal distributed external load; $T(x)$ and $M(x)$ are the shear and moment internal forces; the prime apex means derivative with respect to x; (b) the compatibility equations:

$$\gamma(x) = \varphi(x) + w'(x); \quad \kappa(x) = \varphi'(x) \tag{2a,b}$$

where $w(x)$ and $\varphi(x)$ are the deflection and rotation generalized displacements, while $\kappa(x)$ and $\gamma(x)$ are the curvature and shear angle strains; (c) the constitutive equations:

$$T(x) = t(x)\gamma(x); \quad M(x) = b(x)\kappa(x) \tag{3a,b}$$

where $b(x)$ and $t(x)$ are the moment and shear stiffness. Assuming that these last stiffness quantities are constant along the axis and

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assembling the previous equations, the following two governing equations can be easily obtained:

$$t(\varphi'(x) + w''(x)) = -q(x); \quad b\varphi''(x) = t(\varphi(x) + w'(x)) \tag{4a,b}$$

These are the two classical differential equations used, together with the opportune boundary essential and/or natural conditions, for solving the TBM elastic problem.

3. Alternative single governing differential equations for the TBM

In this section we will show that the elastic problem of the TBM can be governed by a single fourth order differential equation where the unknown variable is the deflection $w(x)$ or a fictitious generalized deflection $\bar{w}(x)$, which is related to the rotation $\varphi(x)$.

Hence, Eqs. (4a,b) are rewritten as:

$$\varphi'(x) = -\frac{q(x)}{t} - w''(x); \quad b\varphi'''(x) = t(\varphi'(x) + w''(x)) \tag{5a,b}$$

Substituting Eq. (5a) and its second order derivative in Eq. (5b) the following fourth order differential equation is obtained:

$$w''''(x) = \frac{q(x)}{b} - \frac{q''(x)}{t} \tag{6}$$

It is interesting to note that this equation differs respect to the elastic EBBM governing equation for the presence of the which the second derivative of the load. Consequently, for constant and linear transversal loads, the governing equations for the EBBM and TBM are coincident.

Nevertheless, substantial differences arise in the imposition of the boundary conditions. As a matter of fact, in the TBM the rotation, the moment and the shear force are expressed in terms of $w(x)$ as follows:

$$\begin{aligned} &M(x) = -bw''(x) - \frac{b}{t}q(x); \quad T(x) = -bw'''(x) - \frac{b}{t}q'(x) \\ &\varphi(x) = -w'(x) - \frac{b}{t}w'''(x) - \frac{b}{t^2}q'(x) \end{aligned} \tag{7a–c}$$

It is important to note that these relationships coincide with those related to the EBBM if the stiffness ratio b/t is set to be zero. Now, the TBM solving equation will be obtained respect to a new fictitious displacement variable, $\bar{w}(x)$. Eqs. (4a,b) can be combined in a single equation in terms of the rotation $\varphi(x)$, that is:

$$b\varphi'''(x) = -q(x) \tag{8}$$

It is well known that a third order differential equation cannot govern the problem of a beam, because of the need to satisfy four boundary conditions. Hence, the fictitious deflection $\bar{w}(x)$ is introduced so as $\varphi(x) = -\bar{w}'(x)$. As a consequence, Eq. (8) is rewritten as:

$$b\bar{w}''''(x) = q(x) \tag{9}$$

that has exactly the same form of the differential equation governing the EBBM. Eq. (9) defines the variable $\bar{w}(x)$ in spite of a constant. Nevertheless, it is simple to verify that the following relationship holds except for a constant that can be chosen equal to zero because it is unessential in the other relationships:

$$\bar{w}(x) = w(x) - \frac{M(x)}{t} \tag{10}$$

In fact, the derivative of each of the members of this equation give the opposite of the rotation. As it will be seen in Section 5, this relationship is fundamental for defining the properties of the proposed approach.

Another very interesting result is that the relationships between $\bar{w}(x)$ and the rotation, the bending moment and the shear force have the same forms of those of the EBBM in terms of $w(x)$; they are:

$$\varphi(x) = -\bar{w}'(x); \quad M(x) = -b\bar{w}''(x); \quad T(x) = -b\bar{w}'''(x) \tag{11a–c}$$

These relationships are completed by that one giving the effective deflection, that can be easily obtained by replacing Eq. (11b) into Eq. (10):

$$w(x) = \bar{w}(x) - \frac{b}{t}\bar{w}''(x) \tag{12}$$

Therefore, it has been proved that the TBM can be solved respect to a single variable, $w(x)$ or $\bar{w}(x)$; namely, both the single differential governing equation and the essential and/or natural boundary conditions can be expressed in terms of this single variable. This idea suggests the definition of a new FE approach for the TBM, that is the fundamental subject of this paper.

4. Some classical finite element approaches

For clarity's sake, in this section we will remind some well-known approaches with the aim of highlighting the main differences with respect to the proposed one. One of the most classical approaches, based on the application of the variational methods, can be found in Reddy (1993). In particular, it considers the weak form of Eqs. (4a,b). The weak form of Eq. (4a) shows that the deflection is a primary variable, whose corresponding secondary variable is the shear force; whereas, in the weak form of Eq. (4b) the primary variable is the rotation and the secondary one is the moment. In each finite element the deflection and the rotation are interpolated by independent Lagrange polynomials that could have different orders. The choice of using the same order of interpolation is very common because it requires two primary variables for any node. But, because of the presence of the derivative operator in the relationship between $w(x)$ and $\varphi(x)$, this choice leads to a numerical problem known as shear locking. To avoid this problem, two alternative procedures have been introduced in the literature (also reported in Reddy (1993)): (a) the Consistent Interpolation Element (CIE) method, that uses for $w(x)$ an interpolation of one order greater than that related to $\varphi(x)$; (b) the Reduced Integration Element (RIE) method, that uses the same order of interpolation, but evaluates the contributions to the stiffness terms depending on the bending deformation energy by the actual interpolation of $\varphi(x)$, while, those depending on the shear deformation energy are evaluated by using one order lesser polynomial. Another interesting approach in the literature is the so-called Interdependent Interpolation Element (IIE) method, introduced by Reddy (1997). This method is based on the solution of the homogeneous equations corresponding to Eqs. (4a,b); it corresponds to applying a third order interpolation on $w(x)$ and implies the presence of only two nodes for finite element. Imposition of the minimum potential energy principle leads to the following expression of the finite element stiffness matrix:

$$\mathbf{k}_t = \frac{2b}{\mu h^3}\begin{bmatrix} 6 & -3h & -6 & -3h \\ -3h & 2h^2\lambda & 3h & h^2\alpha \\ -6 & 3h & 6 & 3h \\ -3h & h^2\alpha & 3h & 2h^2\lambda \end{bmatrix}; \quad \begin{aligned} \alpha &= 1 - 6\Omega \\ \lambda &= 1 + 3\Omega \\ \mu &= 1 + 12\Omega \\ \Omega &= b/\left(th^2\right) \end{aligned} \tag{13}$$

Reddy considered it as the exact stiffness matrix for the Timoshenko beam theory.

5. Proposed method

The starting point of this method is the application of the weak formulation directly to Eq. (9). If a generic finite element of length