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Reissner–Sagoci problem for functionally graded materials with arbitrary spatial variation of material properties

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ABSTRACT

The paper considers the elastostatic problem related to the axisymmetric rotation of a rigid circular punch which is bonded to the surface of a functionally graded coating with arbitrarily varying shear modulus on a homogeneous half-space. The graded coating is modeled as a linear multi-layered medium. By the use of the transfer matrix method and Hankel transform technique, the problem is reduced to a singular integral equation. The stresses at the contacting surface are calculated by solving the equation numerically. The results show that the contact stresses are significantly affected by the gradient of the coating. This allows us to modify the distribution of the contact stress by adjusting the gradient of the coating. The results also provide us a possible way to measure the gradient of the coating using the non-destructive twisting test.

1. Introduction

Functional graded materials (FGMs) are a new kind of nonhomogeneous composites which consist of a gradual change in the volume fraction of constituents from one location to the other. When the FGM is used as a coating, appropriate gradual variation of its elastic modulus may significantly alter the stresses around the indenter and lead to suppression of Hertzian cracking at the edge of the contact region (Giannakopoulos et al., 1997). So control of gradients in mechanical properties offers opportunities for the design of surfaces with resistance to contact deformation and damage that cannot be realized in conventional homogeneous materials (Suresh, 2001). In the past few years, some researchers paid attention to the contact problems of FGMs. The axisymmetric problems of a graded half-space subjected to a concentrated load or to a flat, spherical or conical indenter were considered by Giannakopoulos and Suresh (1997); Giannakopoulos and Suresh (1997). They assumed that the elastic modulus varies in depth in the manner of a power-law function or an exponential function. Guler and Erdogan (2004) have developed a model where the material properties vary as exponential functions and solved the two-dimensional contact problem of a functionally graded coating. Ke and Wang (2006) employed the linear multi-layered model to solve the two-dimensional contact problem of an FGM with an arbitrarily varying shear modulus. Liu et al. (2008) extended the linear multi-layered model to solve the axisymmetric frictionless contact of an FGM. In this paper, the Reissner–Sagoci problem will be considered for the graded coating with arbitrary spatial variation of material properties by using the same model. Reissner and Sagoci (1944) and Sneddon (1947) first considered the problem of torsion of a semi-infinite, isotropic, homogeneous, elastic solid when a circular cylinder was welded to its plane boundary and forced to rotate about its axis. This problem is now referred to Reissner–Sagoci problem. The extension of the Reissner–Sagoci problem to include effects of material non-homogeneity is of interest to geomechanics as well as to non-destructive material testing. The

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torsional indentation of a non-homogenous half space region which exhibits either exponential or power-law variation in the shear modulus was examined by Kassir (1970), Chuapraser and Kassir (1973). A complete and informative account of the torsional indention problems can be found in the monograph by Gladwell (1980).

To take into account arbitrary variation of the material properties and to avoid any discontinuity, we will use the linear multi-layered model (Liu et al., 2008) in this paper. This model is based on the fact that an arbitrary curve can be approached by a series of continuous but piecewise linear curves, and thus divides an FGM into a series of sub-layers with the elastic modulus varying linearly in each sub-layer and continuous on the sub-interfaces. By using the transfer matrix method and Hankel integral transform technique, the problem is reduced to a Cauchy singular integral equation. The stresses at the contacting surface are calculated by solving the equation numerically.

2. Formulation of the problem

Consider the problem shown in Fig. 1a. A rigid circular cylinder adheres to an FGM coating with thickness h_0 which is bonded to a homogeneous half-space. It is assumed that the torque T is applied to the rigid cylinder. The half-space is homogeneous with the shear modulus μ^*. The shear modulus of the functionally graded coating can be described by an arbitrary continuous function of z, $\mu(z)$, with the boundary value $\mu(h_0) = \mu_0$. The linear multi-layered model (Fig. 1b) divides the graded coating into N sub-layers. The shear modulus in each sub-layer is assumed to be the following form:

$$\mu(z) \approx \mu_j(z) = \bar{\mu}_j(a_j + b_j z), \quad h_j < z < h_{j-1}, \quad j = 1, 2, \ldots, N \tag{1}$$

where $\bar{\mu}_j$ is the practical value of the shear modulus at the sub-interface $z = h_j$, i.e.$\bar{\mu}_j = \mu(h_j) = \mu_j(h_j)$. So we have

$$a_j = \frac{h_{j-1} - h_j\bar{\mu}_{j-1}/\bar{\mu}_j}{h_{j-1} - h_j}, \quad b_j = \frac{\bar{\mu}_{j-1}/\bar{\mu}_j - 1}{h_{j-1} - h_j}. \tag{2a, b}$$

Since the system under consideration is axisymmetric, the non-vanishing stress components ($\sigma_{z\theta}$,$\sigma_{r\theta}$) and the equilibrium equation are expressed as

$$\sigma_{z\theta} = \mu(z)\frac{\partial u_\theta}{\partial z}, \quad \sigma_{r\theta} = \mu(z)r\frac{\partial}{\partial r}(\frac{u_\theta}{r}), \quad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{z\theta}}{\partial z} + 2\frac{\sigma_{r\theta}}{r} = 0. \tag{3a, b, c}$$

The equation governing the elastic behavior in each nonhomogeneous layer reads

$$\frac{\partial u_{\theta j}^2}{\partial r^2} + \frac{1}{r}\frac{\partial u_{\theta j}}{\partial r} - \frac{u_{\theta j}}{r^2} + \frac{\partial^2 u_{\theta j}}{\partial z^2} + \frac{1}{\mu_j(z)}\frac{d\mu_j(z)}{dz}\frac{\partial u_{\theta j}}{\partial z} = 0, \quad j = 1, 2, \ldots, N, \tag{4}$$

with $u_{\theta j}$ being the only nontrivial displacement component. The equation in the homogeneous half-space can be written as

$$\frac{\partial u_{\theta,N+1}^2}{\partial r^2} + \frac{1}{r}\frac{\partial u_{\theta,N+1}}{\partial r} - \frac{u_{\theta,N+1}}{r^2} + \frac{\partial^2 u_{\theta,N+1}}{\partial z^2} = 0. \tag{5}$$

The displacements and stresses are continuous at the sub-interfaces, $z = h_j$, which states

$$u_{\theta j}(r, h_j) = u_{\theta,j+1}(r, h_j), \quad 0 \leqslant r < \infty \tag{6}$$

$$\sigma_{z\theta j}(r, h_j) = \sigma_{z\theta,j+1}(r, h_j), \quad 0 \leqslant r < \infty. \tag{7}$$

Along the coating surface, $z = h_0$, we have

$$\sigma_{z\theta 1}(r, h_0) = \tau(r), \quad 0 \leqslant r < a \tag{8}$$

$$\sigma_{z\theta 1}(r, h_0) = 0, \quad r > a, \tag{9}$$

where $\tau(r)$ is the shear stress distribution to be determined.

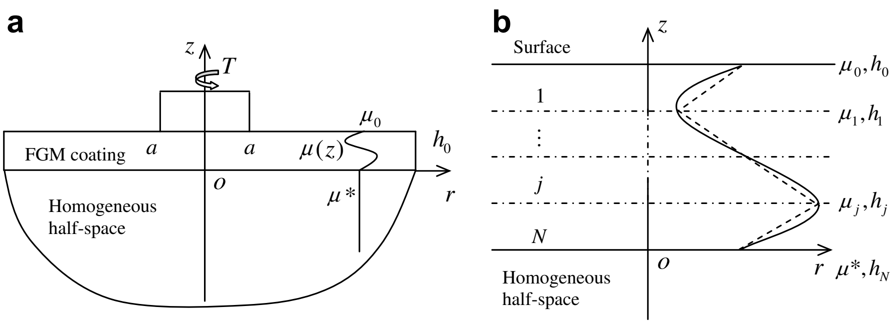

Fig. 1. Functionally graded coated half-space subjected to a torsional load (a) and the linear multi-layered model for the graded coating (b).