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Mechanics Research Communications 33 (2006) 628–635
MECHANICS
RESEARCH COMMUNICATIONS
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A unified asymptotic theory of the anelastic approximation in geophysical gases and liquids

Pierre-Antoine Bois

Laboratoire de Mécanique de Lille, UMR CNRS 8107 UFR de Mathématiques, Bât. M3, U.S.T.L., F-59655 Villeneuve d'Ascq, France

Available online 13 July 2005

Abstract

An asymptotic theory of the anelastic approximation is developed for fluids having arbitrary equations of state under two assumptions: weak compressibility and small Brunt–Väisälä frequency. We show that both Boussinesq approximation (BA) and anelastic approximation (AA) may be included in a unique *quasi-incompressible approximation* (QIA) already constructed by Durran for polytropic gases. The only difference between AA and BA is that, in the BA, the equations are with slowly varying coefficients, while in the AA the coefficients are fast varying. Applications are made to atmospheric air and to sea-water.

Keywords: Anelastic approximation; Boussinesq approximation; Quasi-incompressible approximation; Deep convection; Shallow convection; Weakly compressible fluids

1. Introduction

Deep convection, in geophysical flows, is characterized by (i) a small Mach number and, (ii), a large vertical wavelength (e.g. of the same order of magnitude as the whole height of the troposphere). However a study of degeneracies of Navier–Stokes equations for vanishing Mach number (say ε) and Froude number (say F), leads, in general, to three limits only, namely: *solenoidal flow* ($F^2 < \varepsilon$), *quasi-static flow* ($F^2 > \varepsilon$), and the *Boussinesq approximation*, (BA, limit case $F^2 = \varepsilon$): among these limits, the equations of convection (the BA) depict shallow convection only, so that the deep convection must be looked for by another way. Such way is to take into account the small square of the Brunt–Väisälä frequency, say $N^2(z)$, of the medium.

A model taking into account a small $N^2(z)$, was proposed by Ogura and Phillips (1962), who call it *anelastic approximation* (AA). Its formalism was not mathematically developed, and several papers progres-

E-mail address: pbois@pop.univ-lillel.fr

0093-6413/$ - see front matter
doi:10.1016/j.mechrescom.2005.05.016

sively improved the theory (Lipps and Hemler, 1982; Lipps, 1990). A paper including together the AA and the BA was proposed by Durran (1989) who defined, for polytropic gases, the *quasi-incompressible approximation* (QIA). Formulations of the AA are found, today, in books such as those of Zeytounian (1990), Emanuel (1994), and Durran (1999).

The aim of the present paper is, first, to construct a mathematical support for the AA, and to mathematically relate the AA and the BA. Second, we define the QIA for arbitrary equations of state. In particular, we show that, through the QIA, both BA and AA reduce to the same equations with the only difference that these equations are with slowly varying coefficients for the BA and with fast varying in the AA. An application is, then, made to sea-water.

2. Asymptotic modelling

2.1. Small parameters and degenerate equations

We consider a pure non-dissipative fluid. Written using a classical notation the equations of motion read

$$\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = 0, \qquad \rho\,\mathrm{d}\mathbf{u}/\mathrm{d}t + (1/\varepsilon^2)\{\nabla p + (1/F^{*2})\rho\mathbf{k}\} = 0, \tag{1}$$

$$-(\lambda T\rho/c_p\theta)\mathrm{d}\theta/\mathrm{d}t = \mathrm{d}\rho/\mathrm{d}t - (1/c^2)\mathrm{d}p/\mathrm{d}t = 0, \qquad \rho = f(p,T). \tag{2}$$

$\mathbf{k}$ is the unit vector of vertical direction. The variable θ figuring in the energy equation is the *potential temperature* of the medium, related with the entropy per unit of mass s' by the relation $\mathrm{d}\theta/\theta = \mathrm{d}s'/c_p^*$. The coefficient λ is the thermal expansivity: $\lambda = -(1/\rho)(\partial\rho/\partial T)_p$. The function f defines the equation of state, the sound speed c is known by a function $c^2 = g(p,\rho)$ (say) when f is known. The scaling is made as follows:

$$\begin{cases} \mathbf{u}' = U\mathbf{u}, \quad \mathbf{x}' = L\mathbf{x}, \quad t' = (L/U)t, \quad T' = \Theta^*T, \quad \theta' = \Theta^*\theta, \quad s' = c_p^*\mathrm{Log}\,\theta, \\ \rho' = \rho^*\rho, \quad p' = \rho^*c_p^*\Theta^*p, \quad c_p' = c_p^*c_p, \quad c'^2 = c_p^*\Theta^*c^2, \quad \lambda' = \lambda/\Theta^*, \end{cases} \tag{3}$$

where the primed variables denote physical variables, and U, L, and the starred variables denote the (constant) reference quantities. The two numbers ε and F^* are defined by

$$\varepsilon = U\Big/\sqrt{c_p^*\Theta^*}, \qquad F^* = \sqrt{c_p^*\Theta^*/Lg}. \tag{4}$$

The parameter ε is a compressibility parameter. The number F^{*2} may be interpreted as the ratio of two lengths $H = c_p^*\Theta^*/g$ (length scale associated with the medium), and L (length scale associated with the considered problem). The Froude number F of the flow is nothing but $F = F^*\varepsilon = U/\sqrt{gL}$. We look for degenerate forms of Eqs. (1) for vanishing ε. Hence we set

$$\begin{cases} \mathbf{u} = \bar{\mathbf{u}} = \bar{\mathbf{u}}_0 + \eta_u(\varepsilon)\bar{\mathbf{u}}_1 + \cdots, \quad p = P_0 + \eta_p(\varepsilon)\bar{p}, \quad \rho = R_0 + \eta_\rho(\varepsilon)\bar{\rho}, \\ T = T_0 + \eta_T(\varepsilon)\overline{T}, \quad \theta = \Theta_0 + \eta_\theta(\varepsilon)\bar{\theta}, \quad c = C_0 + \eta_c(\varepsilon)\bar{c}, \end{cases} \tag{5}$$

where the $\eta_u(\varepsilon)$'s, etc. are gauge functions a priori undetermined, but such as $\lim_{\varepsilon\to0}(\eta_u(\varepsilon)) = 0$ etc. We assume that P_0, R_0, T_0, etc. depend on z only. It results that

$$P_0 = P_0(\zeta), \quad R_0 = R_0(\zeta), \quad T_0 = T_0(\zeta), \quad \Theta_0 = \Theta_0(\zeta), \quad C_0 = C_0(\zeta), \quad P_0'(\zeta) + R_0 = 0, \tag{6}$$

where $\zeta = F^{*-2}z$. Now consider the equations at next order with respect to ϵ. After discarding the static terms the remaining momentum equation and the energy equation read

$$R_0(\zeta)\mathrm{d}\bar{\mathbf{u}}_0/\mathrm{d}t + (1/\varepsilon^2)\{\eta_p(\varepsilon)\nabla\bar{p} + (\eta_\rho(\varepsilon)F^{*-2})\bar{\rho}\mathbf{k}\} = 0, \tag{7}$$

$$F^{*-2}R_0(\zeta)N^2(\zeta)\bar{w} = \eta_\rho(\varepsilon)\mathrm{d}\bar{\rho}/\mathrm{d}t - (\eta_p(\varepsilon)/C_0^2(\zeta))\mathrm{d}\bar{p}/\mathrm{d}t - 2\eta_c(\varepsilon)F^{*-2}[R_0(\zeta)/C_0^3(\zeta)]\bar{c}\bar{w}, \tag{8}$$