Mechanics Research Communications 61 (2014) 47–52

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

Fundamental solutions for chiral solids in gradient elasticity

D. Ieşan*

Octav Mayer Institute of Mathematics Romanian Academy, Bd. Carol I, Nr. 8, 700508 Iaşi, Romania

ARTICLE INFO

Article history:
Received 4 June 2014
Received in revised form 15 July 2014
Accepted 22 July 2014
Available online 30 July 2014

Keywords:
Chiral materials
Gradient elasticity
Concentrated loads
Boussinesq–Somigliana–Galerkin solution
Fundamental solutions

ABSTRACT

This paper is concerned with the linear theory of gradient elasticity. The deformation of homogeneous and isotropic chiral materials subjected to concentrated body forces is investigated. First, a counterpart of the Cauchy–Kowalewski–Somigliana solution in the dynamic theory of classical elasticity is established. Then, a general solution of the field equations that is analogous to the Boussinesq–Somigliana–Galerkin solution in the classical elastostatics is presented. The results are used to derive the fundamental solutions of the displacement equations in the equilibrium theory and in the case of steady vibrations.

1. Introduction

In recent years the mechanical behavior of chiral materials has been the subject of many investigations. The deformation of chiral materials is of interest for the investigation of bones, carbon nanotubes, auxetic materials, as well as composites with inclusions. In this paper we use the theory of gradient elasticity (Toupin, 1962; Mindlin, 1964) to establish the fundamental solutions of the field equations for isotropic chiral solids. This work is motivated by the recent interest in using gradient elasticity to model the chiral behavior of elastic materials (see Maranganti and Sharma, 2007; Auffray et al., 2009; Papanicolopulos, 2011; Askes and Aifantis, 2011 and references therein). We note that the gradient elasticity has been recently used to investigate the behavior of carbon nanotubes (Wang and Hu, 2005; Wang and Wang, 2007; Aifantis, 2009; Zhang et al., 2010; Yayli, 2011). Papanicolopulos (2011) established the constitutive equations of isotropic chiral solids in linear gradient elasticity. The field equations show that the chiral behavior is related to the gradient of the rotation.

In the present paper we consider the linear theory of gradient elasticity for homogeneous and isotropic chiral solids. In the case of centrosymmetric materials, Mindlin (1964) established a general solution of the displacement equations of equilibrium and used it to derive fundamental solutions for isotropic solids. In this paper we extend these results to chiral materials. Following Mindlin (1964), we first establish general solutions of the displacement equations for chiral materials. Then, we use these solutions to investigate the effects of the concentrated body forces. In Section 2 we present the basic equations of this theory. Section 3 is devoted to a counterpart of the Cauchy–Kowalewski–Somigliana solution in the dynamic theory of classical elasticity. A general solution of the field equations that is analogous to the Boussinesq–Somigliana–Galerkin solution in the classical elastostatics is also established. In Section 4 we use the representation of solutions given in the preceding section to derive the fundamental solutions of the displacement equations in the equilibrium theory and in the case of steady vibrations. The fundamental solutions play an important role in both applied and theoretical studies on the mechanics of solids. They can be used to construct various analytical solutions of practical problems when boundary conditions are imposed. The fundamental solutions are used in the potential theory (Kupradze et al., 1979; Ieşan, 2009) and they are essential in the boundary element method as well as the study of cracks, defects and inclusions (Sharma, 2004 and references therein). In the case of achiral materials the fundamental solutions in the gradient elastostatics have been established by Mindlin (1964) and Rogula (1973).

* Tel.: +40 232201060; fax: +40 232201160.
E-mail address: iesan@uaic.ro

http://dx.doi.org/10.1016/j.mechrescom.2014.07.006
0093-6413/

2. Basic equations

Mindlin (1964) presented three forms of the linear theory of gradient elasticity. The relations among the three forms have been established by Mindlin and Eshel (1968). In what follows we will use the first form of the gradient elasticity. We note that the three forms of the theory lead to the same displacement-equations of motion of isotropic chiral elastic solids.

In this section we present the fundamental equations of the linear gradient elasticity. Let us consider a body that in the undeformed state occupies the regular region B of euclidean three-dimensional space and is bounded by the surface ∂B. We refer the deformation of the body to a fixed system of rectangular axes Ox_j, $(j = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts (unless otherwise specified) are understood to range over the integers (1,2,3), whereas Greek subscripts to the range (1,2); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate. In all that follows, we use a superposed dot to denote partial differentiation with respect to the time.

We assume that B is occupied by a homogeneous and isotropic chiral elastic solid. Let $\boldsymbol{u}$ be the displacement vector field on B. The strain measures are defined by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ijk} = u_{k,ij}. \tag{1}$$

The potential energy density for a homogeneous and isotropic chiral elastic body is given by (Mindlin and Eshel, 1968; Papanicolopulos, 2011)

$$W = \frac{1}{2}\lambda e_{rr}e_{jj} + \mu e_{ij}e_{ij} + \alpha_1\kappa_{iik}\kappa_{kjj} + \alpha_2\kappa_{ijj}\kappa_{irr} + \alpha_3\kappa_{iir}\kappa_{jjr} + \alpha_4\kappa_{ijk}\kappa_{ijk} + \alpha_5\kappa_{ijk}\kappa_{kji} + 2f\varepsilon_{ikm}e_{ij}\kappa_{kjm}, \tag{2}$$

where ε_{ijk} is the alternating symbol and λ, μ, α_s, $(s = 1, 2, \ldots, 5)$, and f are prescribed constants. The constitutive equations for the stress tensor and double stress tensor are

$$\tau_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + f(\varepsilon_{ikm}\kappa_{jkm} + \varepsilon_{jkm}\kappa_{ikm}),$$

$$\mu_{ijk} = \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + 2\kappa_{krr}\delta_{ij} + \kappa_{rrj}\delta_{ik}) + \alpha_2(\kappa_{irr}\delta_{jk} + \kappa_{jrr}\delta_{ik}) + 2\alpha_3\kappa_{rrk}\delta_{ij} + 2\alpha_4\kappa_{ijk} + \alpha_5(\kappa_{kji} + \kappa_{kij}) + f(\varepsilon_{iks}e_{js} + \varepsilon_{jks}e_{is}), \tag{3}$$

where δ_{ij} is the Kronecker delta.

In the gradient elasticity the equations of motion are expressed in the form

$$\tau_{jk,j} - \mu_{ijk,ij} + \rho F_k = \rho\ddot{u}_k, \tag{4}$$

where F_k is the body force per unit mass and ρ is the density in the reference configuration.

From (1), (3) and (4) we obtain the field equations in terms of the displacement field,

$$\Box_2\boldsymbol{u} + [c_1^2 - c_2^2 - (c_1^2\ell_1^2 - c_2^2\ell_2^2)\Delta]\,\text{grad div}\,\boldsymbol{u} + 2f_0\Delta\,\text{curl}\,\boldsymbol{u} = -\boldsymbol{F}, \tag{5}$$

where we have used the notations

$$\begin{aligned}
&\Box_\alpha = c_\alpha^2(1 - \ell_\alpha^2\Delta)\Delta - \frac{\partial^2}{\partial t^2}, \quad (\alpha = 1, 2),\\
&c_1^2 = (\lambda + 2\mu)/\rho, \quad c_2 = \mu/\rho, \quad \ell_1^2 = \frac{2}{\lambda + \mu}\sum_{j=1}^{5}\alpha_j,\\
&\ell_2^2 = \frac{2}{\mu}(\alpha_3 + \alpha_4), \quad \Delta g = g_{,ii}, \quad f_0 = f/\rho.
\end{aligned} \tag{6}$$

If $f = 0$, then Eq. (5) reduce to those established by Mindlin (1964) for achiral materials. The positive definiteness of the internal energy density implies that (Mindlin and Eshel, 1968; Papanicolopulos, 2011)

$$c_1^2 > c_2^2 > 0, \quad \ell_1^2 > 0, \quad \ell_2^2 > 0, \quad 3f^2 < \mu(2\alpha_4 - \alpha_5).$$

3. General solutions of the field equations

Following Mindlin (1964) we now establish general solutions of the displacement equations. With a view toward deriving fundamental solutions for chiral materials we first establish a solution of the field equations that is analogous to the Cauchy–Kowalewski–Somigliana solution in the dynamic theory of classical elasticity (Gurtin, 1972).

Theorem 1. *Let*

$$\boldsymbol{u} = \Box_1\Box_2\boldsymbol{G} - \{[c_1^2 - c_2^2 - (c_1^2\ell_1^2 - c_2^2\ell_2^2)\Delta]\Box_2 - 4f_0^2\Delta^2\}\,\text{grad}\,\mathit{div}\,\boldsymbol{G} - 2f_0\Box_1\,\text{curl}\,\boldsymbol{G}, \tag{7}$$

where the vector field $\boldsymbol{G}$ *of class* C^{12} *on* $B \times I$ *satisfies the equation*

$$\Box_1(\Box_2^2 + 4f_0^2\Delta^3)\boldsymbol{G} = -\boldsymbol{F}. \tag{8}$$

Then u satisfies the Eq. (5).