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A necessary condition for stability of kinematically indeterminate pin-jointed structures with symmetry

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ABSTRACT

Stability conditions are the key to transform kinematically indeterminate structures into prestressed structures or deployable structures. From the viewpoint of symmetry, a necessary condition is presented for the stability of symmetric pin-jointed structures with kinematic indeterminacy. The condition is derived from the positive definiteness of the quadratic form of the tangent stiffness matrix. Numerical examples verify that the proposed necessary stability condition is in accord with the conventional theory of structural rigidity, and is considered to be more comprehensible. It is robust and easy to implement. Results show that a symmetric prestressed structure is guaranteed to possess integral prestress modes, if the necessary condition is satisfied. Further, a pin-jointed structure with fully symmetric mechanism modes is proved to be unstable, if it does not satisfy the condition.

1. Introduction

Since the first invention of tensegrity structures, various types of kinematically indeterminate pin-jointed structures have been successfully and increasingly applied in many academic as well as engineering fields. They possess strong vitality and have remarkable configurations. Because of the internal mechanisms, these structures could not maintain stable equilibrium states unless proper initial prestresses are introduced. According to whether initial prestresses can stiffen all mechanisms and be contributed to the structural stability, kinematically indeterminate structures are divided into (stable) prestressed structures and (unstable) finite mechanisms. The prestressed structures could be further classified into cable structures, tensegrity structures, cable domes, cable-strut structures, etc. (Tibert and Pellegrino, 2003; Juan and Mirats Tur, 2008; Tran et al., 2012). The finite mechanisms allow significant geometric transformations and can be useful as deployable and foldable structures (Chen et al., 2013). Thus, the stability condition is the key of transforming a kinematically indeterminate structure into a prestressed structure or a deployable structure.

Pellegrino and Calladine (1986) introduced basic concepts for kinematically indeterminate pin-jointed structures. They also presented a linear algorithm to determine whether initial prestress modes would impart first-order stiffness to all internal mechanisms (Calladine and Pellegrino, 1991). The proposed determination criterion has been referred to as the prestress stability condition for pin-jointed structures. Nevertheless, recent studies have validated that the condition is a necessary but not sufficient condition for guaranteeing stable equilibriums (Connelly and Whiteley, 1996; Guest, 2006; Ohsaki and Zhang, 2006). Admittedly, the energy method is a conventional method to study the stability conditions of pin-jointed structures (Motro, 2003; Skelton and de Oliveira, 2009). On the basis of the energy method, Vassart et al. (2000) proposed an analytical method to evaluate the order of internal mechanisms for kinematically indeterminate structures. The method could effectively distinguish finite mechanisms. In fact, general prestressability conditions for kinematically indeterminate pin-jointed structures are difficult to determine (Sultan et al., 2001), as the equilibrium matrix which relates internal force vector with external load vector is singular (Pellegrino and Calladine, 1986), and the stiffness matrices are not always positive definite (Zhang and Ohsaki, 2007). In the field of mathematics, detailed and in-depth investigations on the necessary and sufficient stability conditions for pin-jointed structures have been given (Connelly, 1982; Connelly and Whiteley, 1996; Connelly and Back, 1998; Sultan, 2013). Described in terms of mathematical terminologies and rigidity theory (Zhang et al., 2009), these stability conditions are not easy to understand or implement in numerical ways.

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To overcome this difficulty, some studies have introduced optimization algorithms for investigating the stability of pin-jointed structures, e.g., the genetic algorithm (El-Lishani et al., 2005; Koohestani, 2012, 2013), and the ant colony systems (Chen et al., 2012a,b). These algorithms have powerful heuristic searching ability, however, become inefficient for large-scale structures with a large number of nodes and self-stress states. As many pin-jointed structures are symmetric or periodic, the methods making use of inherent symmetries are efficient and can simplify the complex computation process (Yuan and Dong, 2003; Xi et al., 2011; Tran et al., 2012; Chen and Feng, 2012a). In fact, group theory provides a systematic way to investigate symmetric engineering structures (Altmann and Herzig, 1994). It not only significantly reduces the computation cost (Koohestani, 2011; Chen and Feng, 2012b), but also has a qualitative understanding on the intrinsic properties (Zingoni, 2009; Chen and Feng, 2014).

This study is following the previous work of the authors (Chen et al., 2012a,b), and concentrated on the stability of symmetric pin-jointed structures with internal mechanisms. From the viewpoint of symmetry, we will present a necessary stability condition for symmetric pin-jointed structures. In the symmetry-adapted coordinate system, the stiffness matrices are expressed in the block-diagonalized forms. To guarantee the positive definiteness of quadratic form of the tangent stiffness matrix, a prestressed pin-jointed structure has to hold at least one fully symmetric prestress mode. In other words, members at similar positions keep indistinguishable and equivalent under all the symmetry operations, and they have the same prestress force. The necessary stability condition will be more comprehensible for engineers to understand general stability problems of kinematically indeterminate pin-jointed structures.

2. General stability condition of pin-jointed structures

Here, consider a general kinematically indeterminate pin-jointed structure in d-dimensional Euclidean space (d = 2, 3), which consists of n pin-joints and b members. The pin-jointed structure is said to be stable if it possesses to maintain the equilibrium state. This means it tends to return back to the initial equilibrium state subjected to any small disturbances (displacements). Thus, a stable structure must have the (local) minimality of potential energy at its initial equilibrium state (Connelly, 1982; Connelly and Whiteley, 1996; Ohsaki and Zhang, 2006). According to the Hellinger-Reissner principle, the potential energy Π_R could be expressed as the function of the $nd \times 1$ nodal displacement vector $\boldsymbol{d}$ and the $b \times 1$ Lagrange multiplier vector $\boldsymbol{\Lambda}$, written as:

$$\Pi_R(\boldsymbol{d}, \quad \boldsymbol{\Lambda}) = -\sum_{u=1}^{nd} P_u(X_u - X_u^0) + \frac{1}{2}\sum_{v=1}^{b}\frac{E_v A_v}{l_v^0}(e_v)^2 + \sum_{v=1}^{b} F_v \Lambda_v \tag{1}$$

where P_u is the uth external force, X_u^0 and X_u denote the initial and current coordinates, and the uth entry in the vector $\boldsymbol{d}$ is $\boldsymbol{d}_u = X_u - X_u^0$; E_v, A_v, l_v^0, and e_v denote the Young's modulus, cross-sectional area, initial length, and elastic elongation of the member v, respectively. $F_v = f_v(X_1, \cdots, X_u, \cdots, X_{dn})$ is the vth function describing the constraint conditions for the structure, and Λ_v is the corresponding Lagrange multiplier. Specifically, Λ_v may be taken as the internal axial force of the member v. At the equilibrium configuration, the increment of energy induced by arbitrary virtual displacement vector $\delta\boldsymbol{d}$ is:

$$\Delta\Pi_R = \Pi_R(\boldsymbol{d} + \delta\boldsymbol{d}, \quad \boldsymbol{\Lambda}) - \Pi_R(\boldsymbol{d}, \quad \boldsymbol{\Lambda}) > 0 \tag{2}$$

Through the Taylor expansion, Eq. (2) could be rewritten as:

$$\Delta\Pi_R = \delta\Pi_R + \delta^2\Pi_R + \delta^3\Pi_R + O(\delta^4\Pi_R) > 0 \tag{3}$$

where $\delta\Pi_R$, $\delta^2\Pi_R$, $\delta^3\Pi_R$, and $O(\delta^4\Pi_R)$, respectively, are the first, second, third, and high-order variations of the potential energy. At the equilibrium state, it satisfies:

$$\delta\Pi_R = 0 \quad \text{for all vectors} \quad \delta\boldsymbol{d} \tag{4}$$

In this case, the second-order variation term $\delta^2\Pi_R$ is the general condition to evaluate the structural stability for the structure. Actually, the variation $\delta^2\Pi_R$ would be transformed into the quadratic form of the $nd \times nd$ tangent stiffness matrix $\boldsymbol{K}_T$ of the structure. That is:

$$\delta^2\Pi_R = \delta\boldsymbol{d}^T\boldsymbol{K}_T\delta\boldsymbol{d} > 0 \quad \text{for all} \quad \delta\boldsymbol{d} \tag{5}$$

where $()^{\mathrm{T}}$ is the conjugate transpose. In Eq. (5), the stability determination of the structure at the equilibrium state has been summarized as the positive definiteness problem of the $nd \times nd$ tangent stiffness matrix $\boldsymbol{K}_T$. As the minimum eigenvalue $\lambda_{\min}$ of a real symmetric matrix implies the positive definiteness of the matrix, the stability condition of the structure is given as:

$$\lambda_{\min}(\boldsymbol{K}_T)\begin{cases} > 0 & \text{stable} \\ = 0 & \text{critical} \\ < 0 & \text{unstable} \end{cases} \tag{6}$$

The criteria shown in Eqs. (5) and (6) have been widely adopted as the necessary and sufficient stability condition in structural engineering (Zhang and Ohsaki, 2007; Chen et al., 2012a). Note that further computations are required on high-order variations of Π_R if $\lambda_{\min}(\boldsymbol{K}_T) = 0$, as $\delta^2\Pi_R = 0$ for some $\delta\boldsymbol{d}$ at the critical case (see Eq. (6)). Moreover, rigid-body motions are not considered for free-standing structures in the stability analysis. They could be properly constrained and the corresponding zero eigenvalues in the stiffness matrices are excluded.

The $nd \times nd$ tangent stiffness matrix $\boldsymbol{K}_T$ for a kinematically indeterminate pin-jointed structure is (Guest, 2006; Chen and Feng, 2012b):

$$\boldsymbol{K}_T = \boldsymbol{K}_E + \boldsymbol{K}_G = \boldsymbol{H}\bar{\boldsymbol{G}}\boldsymbol{H}^T + \boldsymbol{K}_G \tag{7}$$

where $\boldsymbol{K}_E$, $\boldsymbol{K}_G$, and $\boldsymbol{H}$ are the $nd \times nd$ modified material stiffness matrix, the $nd \times nd$ geometric stiffness matrix, and the $nd \times b$ equilibrium matrix of the structure, respectively; $\bar{\boldsymbol{G}}$ is a $b \times b$ diagonal matrix containing modified axial stiffness for each member, and its diagonal entry is:

$$\bar{\boldsymbol{G}}_{vv} = \frac{E_v A_v - t_v}{l_v^0} \quad \text{for} \quad 1 \le v \le b \tag{8}$$

where t_v is the initial prestress for the member v. The geometric stiffness matrix can be written as:

$$\boldsymbol{K}_G = ((\boldsymbol{C}_T)^T \cdot \bar{\boldsymbol{t}} \cdot \boldsymbol{C}_T) \otimes \mathbf{I}_d \tag{9}$$

In Eq. (9), C_T is a $b \times n$ connectivity matrix, where the ith and jth entries of the vth row are taken as 1 and −1, and the other entries are 0, if a member v connects the nodes i and j; $\bar{\boldsymbol{t}}$ is a $b \times b$ diagonal matrix, and $\bar{\boldsymbol{t}}_{vv} = t_v/l_v^0$; $\boldsymbol{I}_d$ is a $d \times d$ identity matrix, and $\otimes$ denotes the Kronecker product.

3. Necessary stability condition of symmetric kinematically indeterminate structures

3.1. Matrices expressed in symmetry-adapted coordinate system

Kinematically indeterminate structures have $m \geq 1$ modes of internal infinitesimal or finite mechanism, which bring about computational complications on the structural stability. Denote $\boldsymbol{M}$ as the $nd \times m$ mechanism matrix, and it is obtained from the