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Critical load for buckling of non-prismatic columns under self-weight and tip force

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ABSTRACT

The stability of elastic columns with variable cross-section under self-weight and concentrated end load is considered. A simple and easy-to-implement approach is suggested. Different end conditions are dealt with. The governing equation subject to associated boundary conditions for Euler–Bernoulli columns is transformed into an integral equation, and critical buckling load is then evaluated by seeking the lowest eigenvalue of the resulting integral equation. Numerical examples of the critical buckling load for prismatic and non-prismatic columns under self-weight and end force are given, and the effectiveness of this method for buckling analysis of tapered columns is validated. For several frequently encountered end supports, the influence of the taper ratio on the critical buckling load is discussed.

1. Introduction

Elastic columns are a class of important structural elements, which have been widely used in civil, mechanical, and aerospace engineering. The determination of critical load for buckling of elastic columns is a key problem in engineering design. The first researcher in this field can be traced back to Euler, who pioneered the study of buckling of a prismatic column subjected to a compressive force or under its own weight. Since then, great progress in this field has been made. For example, Gere and Carter (1962) derived exact buckling solutions for several special types of tapered columns with simple boundary conditions in terms of Bessel functions. Siginer (1992) also used the Bessel functions to deal with the buckling of columns with linearly varying inverse of the bending stiffness. For parabolically varying bending stiffness, Ermopoulos (1986) studied the buckling of tapered columns subjected to axially concentrated loads at any position along the length direction. Williams and Aston (1989) further analyzed bounds of the buckling load of tapered columns with certain special second moment of area. With the aid of the Bessel functions, Li (2001) gave a variety of exact solutions for buckling of non-uniform columns under axial concentrated and distributed loading. Using the inverse method, Elishakoff (1999, 2000, 2001) obtained several closed-form solutions for the buckling of inhomogeneous columns with special variable bending stiffness. Furthermore, Li (2009) employed the inverse method and gave exact solutions for the generalized Euler's problem. For a prismatic column under self-weight and tip force, Duan and Wang (2008) exactly determined the buckling load in terms of generalized hypergeometric functions. Recently, Darbandi et al. (2010) put forward to the perturbation method to determine the buckling load of columns with variable cross-section under axial loading. Wang (2010) investigated the stability of a braced standing heavy column and obtained an optimum location of the support for maximum load-carrying capacity. Huang and Li (in press) dealt with buckling of axially graded columns with any axial nonhomogeneity and further gave a suboptimal design of the shape profile of a homogeneous column with constant weight constraint.

Although many methods have been presented to solve the stability of elastic columns with variable cross-section under different boundary conditions, most of them have strict limitation. For instance, application of the special function method such as using Bessel functions strongly depends on the form of an ordinary differential equation with variable coefficients. This paper presents a procedure for determining the buckling load of prismatic and non-prismatic columns under self-weight and tip force.

2. Basic equation

Consider the buckling of a non-prismatic elastic column of length L subjected to an axial compressive force P at its upper tip. When the effect of its own weight is taken into account, we denote distributed axial load as $Q(x)$ along its length direction to describe this effect, where x stands for the axial coordinate measured from the bottom end (Fig. 1). Under such a circumstance, the govern-

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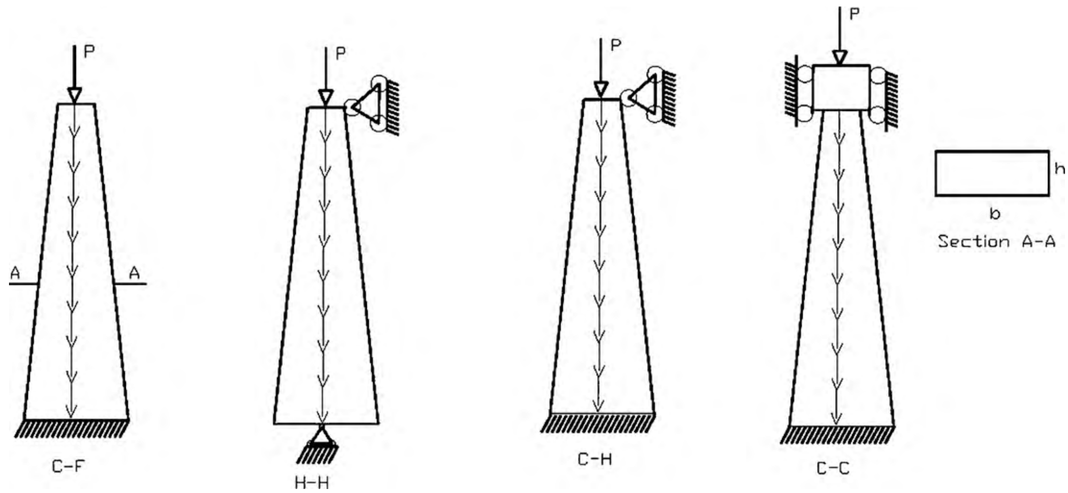

Fig. 1. Schematic of non-prismatic columns under self-weight and tip force.

ing differential equation for elastic buckling of Euler–Bernoulli columns is

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2w}{dx^2}\right]+\frac{d}{dx}\left[P+Q(x)\frac{dw}{dx}\right]=0,\quad 0<x<L, \tag{1}$$

where E is Young's modulus, w is the deflection, and $I(x)$ is the area moment of inertia.

To simplify our analysis, in what follows we introduce the following dimensionless variable: $\xi = x/L$, and consider a non-prismatic column with rectangular cross-section of linearly varying width and thickness, say. Then we have

$$EI(x)=EI_0I(\xi),\qquad Q(x)=q_0Lf(\xi) \tag{2}$$

with

$$I(\xi)=(1-\alpha_1\xi)(1-\alpha_2\xi)^3,\qquad f(\xi)=\int_\xi^1(1-\alpha_1\xi)(1-\alpha_2\xi)\,d\xi, \tag{3}$$

where α_1, α_2 are two taper ratios with respect to the width and thickness directions, respectively, and q_0 denotes weight per unit length. Moreover, $0 \le \alpha_j \le 1$. In particular, when $\alpha_j = 1$, the rectangular column tapers to a sharp tip. This case may make the critical buckling load of elastic columns vanish (Timoshenko and Gere, 1961), and is nearly a theoretical limit because it can never be reached in practice. Consequently, Eq. (1) can be transformed into a normalized form

$$\frac{d^2}{d\xi^2}\left[I(\xi)\frac{d^2w}{d\xi^2}\right]+\lambda_p\frac{d^2w}{d\xi^2}+\lambda_q\frac{d}{d\xi}\left[f(\xi)\frac{dw}{d\xi}\right]=0, \tag{4}$$

where

$$\lambda_p=\frac{PL^2}{EI_0},\qquad \lambda_q=\frac{q_0L^3}{EI_0}. \tag{5}$$

Now we integrate both sides of Eq. (4) four times and then get

$$I(\xi)w(\xi)+\int_0^\xi K(\xi,s)w(s)\,ds=\frac{C_1}{6}\xi^3+\frac{C_2}{2}\xi^2+C_3\xi+C_4 \tag{6}$$

where C_j ($j = 1, 2, 3, 4$) are unknown constants to be determined through boundary conditions at both ends, and the kernel function $K(\xi, s)$ is

$$K(\xi,s)=-2I'(s)+(\xi-s)[I''(s)+\lambda_p+\lambda_qf(s)]-\frac{\lambda_q}{2}(\xi-s)^2f'(s). \tag{7}$$

In the above, the prime represents differentiation with respect to the argument.

3. Solution procedure

Here consider several typical heavy columns under an end force. For clamped-free (C-F) columns with clamped end $x = 0$ and free end $x = L$, the boundary conditions for this case can be written below

$$w(0)=0,\qquad w'(0)=0, \tag{8}$$

$$\frac{d^2w}{d\xi^2}=0,\qquad \frac{d}{d\xi}\left[I(\xi)\frac{d^2w}{d\xi^2}\right]+\lambda_p\frac{dw}{d\xi}=0,\qquad \xi=1. \tag{9}$$

Using the above conditions, we easily find

$$C_1=0,\quad C_2=\frac{-2}{2I(1)-\lambda_p}\int_0^1[\lambda_pK(1,s)+\lambda_qI(1)f'(s)]w(s)\,ds.$$
$$C_3=0,\quad C_4=0$$

After substitution of the above results of C_j into Eq. (6), we get an integral equation with respect to w as follows:

$$I(\xi)w(\xi)+\int_0^\xi K(\xi,s)w(s)\,ds+\int_0^1 H(\xi,s)w(s)\,ds=0, \tag{10}$$

where

$$H(\xi,s)=\frac{\xi^2}{2I(1)-\lambda_p}[\lambda_pK(1,s)+\lambda_qI(1)f'(s)]=0. \tag{11}$$

Using the same procedure, for other elastic columns with the following boundary conditions:

$w=0,\quad w'=0\quad \text{at}\quad \xi=0,1,$

for clamped-clamped (C-C) columns;

$w=0,\quad w''=0\quad \text{at}\quad \xi=0,1,$ for hinged-hinged (H-H) columns;

$w(0)=w'(0)=0,\quad w(1)=w''(1)=0,$

for clamped-hinged (C-H) columns;

we similarly obtain an integral equation for each case, which is still expressed by (10), but with the following kernel

$$H(\xi,s)=\begin{cases}-\xi^3[K'_\xi(1,s)-2K(1,s)] & \\ -\xi^2[3K(1,s)-K'_\xi(1,s)] & \text{for C-C columns,}\\ \frac{1}{6}\xi^3\lambda_qf'(s)-\xi[K(1,s)+\frac{\lambda_q}{6}f'(s)] & \text{for H-H columns,}\\ \frac{\xi^3-\xi^2}{4}\lambda_qf'(s)+\frac{\xi^3-3\xi^2}{2}K(1,s) & \text{for C-H columns.}\end{cases} \tag{12}$$

where $K'_\xi(\xi,s)=\partial K(\xi,s)/\partial\xi$.

As a consequence, it suffices to determine the eigenvalues of the resulting integral Eq. (10). The critical buckling load is in fact related