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Simplified theory and analytical solution for functionally graded thin plates with different moduli in tension and compression

Xiao-Ting He[a,b,*], Xin-Xin Pei[a], Jun-Yi Sun[a,b], Zhou-Lian Zheng[a,b]

[a] School of Civil Engineering, Chongqing University, Chongqing 400045, PR China
[b] Key Laboratory of New Technology for Construction of Cities in Mountain Area, Chongqing University, Ministry of Education, Chongqing 400030, PR China

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ABSTRACT

In this study, a simplified theory for functionally graded thin plates with different moduli in tension and compression is proposed. Based on the classical Kirchhoff hypothesis, a mechanical model concerning tension-compression subzone is established, first. Using the geometrical and physical relations and equation of equilibrium, all stress components are expressed in terms of the deflection, in which modulus of elasticity in tensile and compressive zone are regarded as two different functions while Poisson's ratios are taken as two different constants. Via the equilibrium conditions and continuity conditions, the governing equation expressed in terms of the deflection as well as the unknown neutral layer are derived, respectively. Moreover, the application in polar coordinates, the strain energy and the perturbation solution for the unknown neutral layer, are discussed in detail. The results indicate that the bending stiffness derived in this study play an important role while contacting the classical problem and this problem. The analytical solutions from equilibrium conditions and continuity conditions are consistent. Analyses of more general cases for modulus of elasticity and Poisson's ratio also show the applicability of the simplified theory. This study provides a theoretical basis for the subsequent work.

1. Introduction

Lamination is the technique of manufacturing a material in multiple layers, so that the composite material achieves improved strength, stability, sound insulation, appearance or other properties from the use of differing materials. Compared to conventional laminated materials, functionally graded materials (FGMs) is a kind of material which has innovative ideas. The concept of FGMs can be traced back to the eighties and nineties of last century, and at that time, to eliminate interface problems and relieve thermal stress concentrations in conventional laminated materials, a group of Japanese scientists suggested using this material as thermal barrier materials for aerospace structural applications and fusion reactors [1]. Generally, FGMs is a kind of inhomogeneous composites from the point of macroscopic view that are typically made from a mixture of two materials. This mixture can be obtained by gradually changing the composition of the constituent materials (along the thickness direction of components in most cases). The characteristics of FGMs vary gradually with the thickness direction within the structure, which eliminates interface problems and thus the stress distributions are smooth. Moreover, FGMs possess many new properties that most traditional laminated materials do not have, which makes the use of FGMs has many advantages in aerospace, automotive and biomedical applications. During the past decades, FGMs have received a significant amount of attention from academic community and engineering field, in which the static bending problem of FGMs plates is one of the basic studied problems.

Many scholars have carried out the research on FGMs plates. Reddy et al. [2] examined the axisymmetric bending of functionally graded circular and annular plates by developing exact relationships between the solutions of the classical plate theory and the first-order shear deformation plate theory. Cheng and Batra [3] used an asymptotic expansion method to analyze the isotropic FGMs elliptic plate with clamped edges based on three-dimensional elasticity theory. Ma and Wang [4] studied nonlinear bending and postbuckling of functionally graded circular plates under mechanical and thermal loadings. Based on the Soldatos plate theory, Bian et al. [5] obtained analytical static solutions of single- and multi-span orthotropic FGMs plates subjected to cylindrical bending. Based on the classical plate theory, Chi and Chung [6] presented analytical solutions for simply supported isotropic FGMs plates. Using the stress function method, Li et al. [7] obtained elasticity solutions for transversely isotropic FGMs circular plates subjected

* Corresponding author. Fax: +86 23 65123511.
E-mail address: hexiaoting@cqu.edu.cn (X.-T. He).

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to pure bending. Yang et al. [8] adopted the expansion formula for displacements to obtain elasticity solutions for functionally graded plates in cylindrical bending. Naderi and Saidi [9] studied the pre-buckling configuration of the functionally graded Mindlin rectangular plates whose mechanical properties vary through the thickness. Using the differential reproducing kernel interpolation, Wu et al. [10] developed a meshless collocation method for the plane problems of functionally graded material beams and plates under transverse mechanical loads. Studies on FGMs plates involve many topics including static and dynamic problems, stability, post-buckling, geometrical nonlinear problems and so on. The theory adopted covers many aspects such as the classical plate theory, first order shear and higher order shear theories. In the analysis of FGMs, however, there is no more accurate consideration of the mechanical properties of the materials, especially with respect to stress state in tension and compression.

In this study, we will study another important property of materials, which may be ignored more or less in existing work, i.e., differently mechanical properties in tension and compression. Many studies have indicated that most materials may exhibit different elastic responses in the state of tension and compression. Materials that have apparently different moduli in tension and compression are known as bimodular materials [11], for example, ceramics, graphite, concrete, and some biological materials (nacre, for example Ref. [12]). During the process of the analysis, however, this important characteristic is often neglected due to the complexity from material models. During recent decades, many scholars have proposed some useful material models for studying bimodular materials. One is Bert's model [13] based on the criterion of positive-negative signs of the strains in the longitudinal fibers. This model is widely used in laminated composites [14–17]. Another is Ambartsumyan's bimodular model [18] for isotropic materials. This model assesses different moduli in tension and compression based on the positive-negative signs of principal stresses, which is especially important for the analysis and design of structures. It is well-known that in structural analysis of a building, the cracking direction of a concrete beam is always normal to the direction of principal tensile stresses in the beam. However, if Ambartsumyan's bimodular model is adopted, the stress state of any point in a structure is not correctly indicated beforehand. With the exceptions of some fundamental problems, acquiring the states of the stresses in a structure relies only upon finite element modeling analysis [19–22], which further increases the difficulty in solving bimodular problems.

More recently, some analytical studies of bimodular beams [23,24] and plates [25,26] have been performed. Among these works, the judgment of unknown neutral layer is a key issue because it opens up possibility for the establishment of mechanical model based on subzone in tension and compression. Yao and Ye [24] firstly proposed that the position of the neutral axis in a beam depends only upon the bending stress acting on the cross section, with the shearing stress having no influence on the neutral axis. He et al. [25] used the classical Kirchhoff hypothesis to assess the existence of the elastic neutral layers of a thin plate during bending with a small deflection. Consequently, a series of analytical solutions of plates is derived in rectangular and polar coordinate systems [25,26]. In all of the above studies, however, functional gradient characteristics of the materials are not incorporated. At the same time, to the knowledge of the authors, there are few studies regarding the bending response of a functionally graded plate with bimodular effect. An accurate analysis of structures is not only reflected in the advanced nature of the theory adopted, but also depends on the comprehensive consideration of different properties of materials. Therefore, for the accurate analysis of FGMs plates it is important to incorporate the bimodular effect of materials, in which the mechanical model based on subzone in tension and compression is still valid or not, is a primary problem.

In this study, the bending problem of a functionally graded thin plate with different moduli in tension and compression is analyzed theoretically. This paper is organized as follows. Based on the classical Kirchhoff hypothesis, a mechanical model of subzones in tension and compression is established, first, in Section 2. The three basic equations including geometrical relation, physical relation and equation of equilibrium are given and the stress components are expressed in terms of the deflection in Section 3. Via the equilibrium conditions and continuity conditions, the governing equation and the location of the neutral layer are derived in Section 4, respectively. Moreover, in Section 5 the application in polar coordinates, the strain energy of thin plates, the perturbation solution of the neutral layer, and more general cases for modulus of elasticity and Poisson's ratios are discussed further. In Concluding remarks, the main results are elucidated and the subsequent work is pointed out.

2. The mechanical model

In the small-deflection bending of thin plates, let the stress components be σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} and τ_{zx} and the strain components be ε_x, ε_y, ε_z, γ_{xy}, γ_{yz} and γ_{zx}. The components of displacement at a point, in the x and y directions, are denoted by u and v, respectively. Due to lateral loading, deformation takes place; any point at the midsurface of the plate has deflection w. From Kirchhoff hypotheses, we deduce the two-dimensional stress-strain relation

$$\begin{cases} \varepsilon_x = \dfrac{1}{E(z)}(\sigma_x - \mu\sigma_y), \\ \varepsilon_y = \dfrac{1}{E(z)}(\sigma_y - \mu\sigma_x), \\ \gamma_{xy} = \dfrac{2(1+\mu)}{E(z)}\tau_{xy}, \end{cases} \tag{1}$$

and

$$(\varepsilon_x)_{z=0} = 0, (\varepsilon_y)_{z=0} = 0, (\gamma_{xy})_{z=0} = 0, \tag{2}$$

in which due to the introduction of functionally graded materials, the modulus of elasticity $E(z)$ is not a constant but a function with respect to the thickness direction z, and $E(z)|_{z=0} \neq 0$ holds; the Poisson's ratio μ is regarded as a constant. Eq. (2) says that the midsurface remains unstrained subsequent to bending, i.e., there is no normal strain and no shear strain. Therefore, we may think of the midsurface as the neutral layer of thin plates in bending. From Eqs. (1) and (2), we have

$$\begin{cases} (\sigma_x)_{z=0} = \mu(\sigma_y)_{z=0}, \\ (\sigma_y)_{z=0} = \mu(\sigma_x)_{z=0}, \\ (\tau_{xy})_{z=0} = 0. \end{cases} \tag{3}$$

Substituting the second expression of Eq. (3) into the first one, we have

$$(\sigma_x)_{z=0} = \mu^2(\sigma_x)_{z=0}. \tag{4}$$

Under the condition $(\sigma_x)_{z=0} \neq 0$, $\mu^2 = 1$ may be obtained which fails to satisfy the physical meaning. Only under the condition $(\sigma_x)_{z=0} = 0$, Eq. (4) holds. Similarly, we also obtain $(\sigma_y)_{z=0} = 0$. Therefore, $(\sigma_x)_{z=0} = 0$ and $(\sigma_y)_{z=0} = 0$ are the conditions which the stress components should satisfy at the neutral layer.

The above conclusion is obtained for functionally graded materials based on Kirchhoff hypotheses, and does not account for different moduli in tension and compression of the materials. From the viewpoint of phenomenalism, a functionally graded thin plate with bimodular effect will uniformly generate a deflected shape