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A survey of weakly-nonlinear acoustic models: 1910–2009

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ABSTRACT

A chronological survey of the weakly-nonlinear acoustic equations put forth between the years 1910–2009 is presented. The derivation, properties, and historical development of these models, along with the approximation scheme on which they are based, are discussed. Connections between the models are also discussed and errors/omissions in the literature are noted and, in some cases, corrected. Brief chronologies of the weakly-nonlinear models that describe propagation in relaxing fluids, bubbly liquids, and fluids that saturate porous solids are also presented.

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Contents

1. Introduction

Broadly speaking, the branch of fluid mechanics known as acoustics[1] can be defined as 'the study of irrotational compressible flow.' Like most other areas of physics, the field of acoustics can be broken down into experimental and theoretical sub-fields. In turn, the latter can, itself, be divided into two disciplines, namely, that in which the equations considered are linear, and that in which they are not.

If the signal strengths involved are relatively weak and the propagation distances of interest are relatively short, then the inherently nonlinear system of equations that describe sound propagation can, often to rather good accuracy, be approximated by linear partial differential equations (PDE)s; this, of course, is the realm of linear acoustics, on which essentially all of classical acoustics is based.

In contrast, when the problems of interest involve 'finite-amplitude' [29] acoustic signals and/or extreme propagation distances, simple linear theory usually proves to be inadequate. This occurs because the effects of nonlinearity, being both present in the fundamental equations of fluid flow and cumulative in nature, rapidly become felt over time and distance. As such, it would appear that we are compelled, when confronted with such problems, to set aside our simple linear models in favor of their *fully nonlinear*, and thus more challenging, counterparts. There is, however, another possibility: the so-called *weakly-nonlinear* modeling approach. By this we mean the derivation of approximate equations of motion, which are based on the 'small, but finite-amplitude' (i.e., small Mach number) assumption, from the irrotational Euler and Navier–Stokes–Fourier systems that, while relatively tractable

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[1] Anticipating push-back from some readers, we regard the study of longitudinal waves in solids as an area of *elasticity theory*.

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from the mathematical standpoint, still capture the salient nonlinear phenomena exhibited by compressible flows.

The aim of this communication is to conduct a historical review of these weakly-nonlinear acoustic models, in particular, their formulation, historical development, and relationship to each other, over the century spanning the years 1910–2009. Our primary focus shall be on propagation in single phase, single species[2] compressible fluids flowing in domains that are free of scattering bodies. To this end, we first present, in Section 2, a review of both the fundamental equations governing compressible flow and the major approximations of weakly-nonlinear theory. This is followed, in Section 3, by a detailed chronology of the weakly-nonlinear model equations of what can be called 'traditional' acoustic problems. Lastly, in Section 4, chronologies of the weakly-nonlinear models that describe propagation in relaxing fluids, bubbly liquids, and fluids that saturate porous solids are presented, and aspects of these models are briefly discussed.

2. Acoustic propagation in fluids: review of fundamental equations

2.1. Lossless theory: the Euler equations

Assuming the absence of all external body forces and heat sources, the *homentropic*[3] flow of a lossless fluid is described by the following system of first order PDEs known as *Euler's equations*:

$$\frac{D\varrho}{Dt} = -\varrho(\nabla\cdot\mathbf{v}), \tag{1}$$

which is the *continuity equation*;

$$\varrho\frac{D\mathbf{v}}{Dt} = -\nabla\wp, \tag{2}$$

the *momentum equation*; and

$$\frac{D\eta}{Dt} = 0 \quad (\nabla\eta = \mathbf{0}), \tag{3}$$

the *entropy production equation*. Here, $\mathbf{v} = (u, \upsilon, w)$ is the velocity vector, $\varrho(>0)$ is the mass density, $\wp(>0)$ is the thermodynamic pressure, η is the specific entropy, and D/Dt denotes the material derivative operator.

To close our system, an equation of state (EoS) must be specified. In the case of a *perfect gas*[4] under homentropic flow, the general EoS for a perfect gas (see Eqs. (19) below) reduces to the 'adiabatic law' [51, p. 478]

$$p = -\wp_0\left[1 - (\varrho/\varrho_0)^{\gamma}\right]; \tag{4}$$

see also Refs. [57, §1.1.1] and [72, §1-4]. Here and below, $p := \wp - \wp_0$ denotes the *acoustic* (or over) pressure; $\gamma = c_p/c_v$ is the adiabatic exponent (or index), where we observe that $\gamma \in (1, 5/3]$ in the case of perfect gases; and a zero subscript attached to a quantity denotes the (constant) equilibrium state value of that quantity.

In the case of liquids, on the other hand, the situation is more complicated; see Refs. [57, §1.1.2] and [86, §2.6]. One approach has been to assume a *polytropic law* (i.e., one similar in form to Eq. (4)), as in the case of what is known today as *Tait's EoS*

$$\wp = B_1(\varrho/\varrho_0)^{\Gamma} - B_2. \tag{5}$$

Batchelor [4, §1.8] notes that Eq. (5) is in close agreement with experimental data for water over a wide range of pressures, in particular, those encountered in the deep ocean. Here, the exponent $\Gamma(>1)$, which we take to be constant, has been found to be $\Gamma \approx 7$ in the case of water, and $B_{1,2}$, also regarded as constants,[5] are determined by experimental measurements; see also Refs. [6], [57, §1.1.4], as well as [86, p. 102].

It is noteworthy that Eqs. (4) and (5) are functions of *only* the mass density; i.e., they are *barotropic* [86, p. 56] relations, as theory indicates they must be when the flow in question is homentropic.

If we limit our focus to acoustic propagation, meaning that the flow is taken to be irrotational, it follows that $\mathbf{v} = \nabla\phi$, where ϕ is the *scalar velocity potential*. Thus, on taking notice of the fact that the homentropic assumption also means that η is everywhere and always equal to its equilibrium state value, and employing Ref. [35, Eq. (60.33*a*)] to recast the left-hand side (LHS) of Eq. (2), Sys. (1)–(3) is reduced to

$$s_t = -\nabla\cdot[(1+s)\nabla\phi], \tag{6}$$

$$(1+s)\nabla\left(\phi_t + \frac{1}{2}|\nabla\phi|^2\right) = -c^2\nabla s \quad (\nabla\times\mathbf{v} = \mathbf{0}), \tag{7}$$

$$\eta = \eta_0, \tag{8}$$

while Eqs. (4) and (5) become

$$p = -\wp_0\left[1 - (1+s)^{\gamma}\right], \tag{9}$$

$$\wp = B_1(1+s)^{\Gamma} - B_2, \tag{10}$$

respectively, where $s = (\varrho - \varrho_0)/\varrho_0$ is known as the *condensation*. Here, the speed of sound c, which is a thermodynamic variable, is given by [86, §4.3]

$$c = \sqrt{\partial\wp/\partial\varrho}, \tag{11}$$

an expression which we stress is valid for both gases and liquids [36, p. 999]. In the case of a perfect gas, however, Eq. (11) yields the more explicit result $c = \sqrt{\gamma\wp/\varrho}$, which we observe also holds in the more general case of *ideal* gases [86, p. 165]. If it is also true that the flow in question is homentropic, then one can, with the aid of Bernoulli's theorem [35, §48], recast this (latter) expression for c as [38, §3.2]

$$c^2 = c_0^2 - (\gamma - 1)\left(\phi_t + \frac{1}{2}|\nabla\phi|^2\right), \tag{12}$$

where, in this communication, $c_0(>0)$ denotes the speed of sound in the undisturbed *fluid* [72, §1-9]. In the case of a perfect gas, c_0 is of course given by

$$c_0 = \sqrt{\gamma\wp_0/\varrho_0}. \tag{13}$$

2.2. Thermoviscous theory: the Navier–Stokes–Fourier system

Now introducing the transport coefficients $\mu(>0)$, $\mu_B(\geq 0)$, and $K(>0)$, all of which we regard as constant, and again assuming the absence of all external body forces and sources of heat, Sys. (1)–(3) is generalized to what Pierce [72, §10-1] calls the *Navier–Stokes–Fourier* (NSF) *system*:

$$\frac{D\varrho}{Dt} = -\varrho(\nabla\cdot\mathbf{v}), \tag{14}$$

$$\varrho\frac{D\mathbf{v}}{Dt} = -\nabla\wp + \mu\nabla^2\mathbf{v} + \left(\frac{1}{3}\mu + \mu_B\right)\nabla(\nabla\cdot\mathbf{v}), \tag{15}$$

$$\varrho\vartheta\frac{D\eta}{Dt} = K\nabla^2\vartheta + \Phi, \tag{16}$$

[2] And, of course, mixtures of fluids that can been modeled as such; e.g., air.

[3] Howarth [39, p. 3] credits M.P. Charlesworth with conceiving this term to describe the special case of isentropic flow defined by Eq. (3); see also Thompson [86, p. 60].

[4] By which we mean in the sense of Thompson [86, §2.5]; specifically, an ideal gas (i.e., a gas that obeys Eq. (21) below) in which $c_p > c_v > 0$, the specific heats at constant pressure and volume, respectively, are constants.

[5] In general, $B_{1,2}$ are both slowly varying functions of η [4, §1.8].