Mechanics Research Communications 73 (2016) 58–62

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

# Surface tension-induced stress concentration around an elliptical hole in an anisotropic half-plane

Ming Dai [a,b], Peter Schiavone [b,*], Cun-Fa Gao [a]

[a] State Key Laboratory of Mechanics and Control of Mechanical Structures, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China
[b] Department of Mechanical Engineering, University of Alberta, Edmonton, Alberta T6G 1H9, Canada

ARTICLE INFO

*Article history:*
Received 1 December 2015
Received in revised form 12 February 2016
Accepted 13 February 2016
Available online 21 February 2016

*Keywords:*
Surface tension
Anisotropic material
Half-plane
Nanosized hole
Size dependence

ABSTRACT

We examine the surface tension-induced stress concentration around an elliptical hole inside an anisotropic half-plane with traction-free surface. Using conformal mapping techniques, the corresponding complex potential in the half-plane is expressed in a series whose unknown coefficients are determined numerically. Our results indicate that the maximum hoop stress around the hole (which appears in the vicinity of the point of maximum curvature) increases rapidly with decreasing distance between the hole and the free surface. In particular, for an elliptical or even circular hole in an anisotropic half-plane we find that, with decreasing distance between the hole and the free surface, the hoop stress can switch from compressive to tensile at certain points on the hole's boundary and from tensile to compressive at others. This phenomenon is absent in the case of an elliptical or even circular hole in the corresponding case of an isotropic half-plane.



## 1. Introduction

The modeling and analysis of an elastic half-plane containing holes or inclusions is well-known to have theoretical and practical applications in engineering science. For example, in integrated circuits, the passivated metallic line between the passivation layer and the underlying substrate may be modeled more realistically as a thermal inclusion in a half-plane rather than in a whole plane since the passivation layer is normally much thinner than the substrate [1]. Considerable research is available in the literature concerning the problem of an elastic half-plane with circular hole(s) [2–6], circular inclusion(s) [7–10], elliptical inclusions [11], non-elliptical inclusions [12] and even an arbitrarily-shaped inclusion inside an isotropic or anisotropic half-plane [1,13]. We mention here that the results in Refs. [1,13] are restricted to the case when the inclusion has the same material properties as the surrounding matrix (essentially to prevent the inclusion from degenerating into a hole). The analysis of structures containing holes or inclusions located near the edge of the half-plane is also of particular interest since the proximity of the hole or inclusion to the edge is known to affect the stress distribution around the hole or inclusion.

Recently, considerable attention has been given to the study of materials and structures containing nano-holes or nano-inclusions mainly because of their unique physical and mechanical properties. It is well-known that as the dimension of a hole or inclusion decreases toward the nanoscale, surface/interface effects induced by the contributions of surface/interface energy and surface/interface tension (usually neglected at higher order length scales) can make a substantial contribution to the elastic fields surrounding the holes or inclusions. Consequently investigations into problems involving nano-holes or nano-inclusions inside an elastic half-plane have indeed incorporated surface/interface effects into the model of deformation but most have been restricted to simple cases where the hole or inclusion is circular (see, for example, Refs. [14–16]) with only a few dealing with the non-circular case (see, for example, Refs. [17,18]) and all are limited to the case of isotropic materials. To the authors' knowledge, the analysis of problems involving holes or inclusions in an *anisotropic* half-plane subjected to plane deformations remains absent from the literature even in the simpler case when surface/interface effects are ignored. In this paper, we present an efficient method to deal with the problem of stress concentration around an elliptical hole in an anisotropic half-plane subjected to plane deformations when surface tension is included on the boundary of the hole. This assumption is reasonable for problems involving small strain. In fact, in certain technological applications, the effect of deformation-dependent surface elasticity has been found to be small compared to that of surface tension [19,20].

The paper is organized as follows. The mathematical model is established in Section 2 with corresponding boundary conditions

* Corresponding author. Tel.: +1 780 492 3638.
*E-mail address:* P.Schiavone@ualberta.ca (P. Schiavone).

http://dx.doi.org/10.1016/j.mechrescom.2016.02.010
0093-6413/

described in terms of two complex potentials defined in the half-plane. In Section 3, conformal mapping techniques are used in conjunction with Fourier expansion methods to determine the complex potentials of the half-plane. Various numerical examples are given in Section 4 to verify the accuracy of the present method and also to demonstrate the influence of the half-plane's free surface on the surface tension-induced stress concentration around circular and elliptical holes in the half-plane. Finally, the main results are summarized in Section 5.

## 2. Mathematical model

As shown in Fig. 1, we consider an anisotropic (lower) half-plane containing an elliptical hole whose boundary is described by the curve $L$. The elliptical hole is centered at $y_0$ on the negative $y$-axis and has semi-major and semi-minor axes denoted by $a$ and $b$, respectively. The major axis is assumed to be inclined at an angle $\alpha$ to the positive $x$-axis. As noted above, we consider only the contribution of surface tension and disregard the deformation-dependent surface elasticity on the boundary of the hole. Consequently in our present problem, the edge $L'$ of the half-plane is traction-free while on the hole's boundary $L$, the traction $\sigma_{nn}$ (along the normal to the curve $L$), induced by the constant surface tension $T$, is described by

$$\sigma_{nn} = T\frac{d\beta}{ds}, \quad \text{on}\, L, \tag{1}$$

where $d\beta/ds$ represents the curvature of $L$; $\beta$ (shown in Fig. 1) is the angle between the positive $x$-axis and the tangent to the curve $L$ and $ds$ is the arc length of an element of $L$ along its tangent. With the boundary conditions on $L$ in mind, we rewrite Eq. (1) as

$$\begin{aligned} Y_L &= T\frac{d\beta}{ds}\cos\beta, \quad \text{on}\, L, \\ X_L &= -T\frac{d\beta}{ds}\sin\beta, \quad \text{on}\, L, \end{aligned} \tag{2}$$

where $X_L$ and $Y_L$ are, respectively, the $x$- and $y$-components of the traction on $L$.

The stress distribution in the anisotropic half-plane can be expressed in terms of two complex potentials $\varphi_i(z_i)$ $(z_i = x + \mu_i y, i = 1, 2)$ by [21]

$$\begin{aligned} \sigma_{xx} &= 2Re[\mu_1^2\varphi_1'(z_1) + \mu_2^2\varphi_2'(z_2)], \\ \sigma_{yy} &= 2Re[\varphi_1'(z_1) + \varphi_2'(z_2)], \\ \sigma_{xy} &= 2Re[\mu_1\varphi_1'(z_1) + \mu_2\varphi_2'(z_2)], \end{aligned} \tag{3}$$

where $\mu_1$ and $\mu_2$ are two distinct complex roots with positive imaginary parts with each determined by the following fourth-order equation in $\mu$,

$$a_{11}\mu^4 - 2a_{13}\mu^3 + (2a_{12} + a_{33})\mu^2 - 2a_{23}\mu + a_{22} = 0, \tag{4}$$

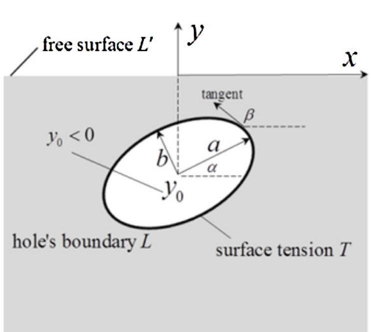


**Fig. 1.** Anisotropic half-plane containing an elliptical hole with surface tension.

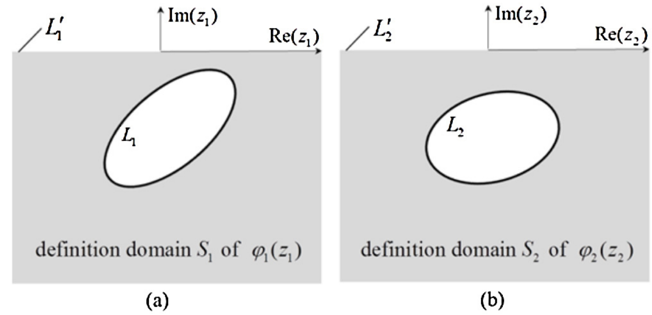


**Fig. 2.** Domain of definition of the complex potentials in the half-plane.

Here, the constants $a_{ij}$ $(i, j = 1, 2, 3)$ are the compliance coefficients of the anisotropic material occupying the half-plane. Fig. 2 shows the domains of definition of the two complex potentials $\varphi_1(z_1)$ and $\varphi_2(z_2)$ in which the curves $L_1$ in the $z_1$-plane and $L_2$ in the $z_2$-plane both correspond to the hole's boundary $L$ in the $xy$-plane while the edges $L_1'$ in the $z_1$-plane and $L_2'$ in the $z_2$-plane both correspond to the free surface $L'$ in the $xy$-plane.

The traction boundary conditions on the hole's boundary $L$ and the free surface $L'$ are described in terms of $\varphi_1(z_1)$ and $\varphi_2(z_2)$ as [21]

$$\left.\begin{aligned} &2Re[\varphi_1(z_1) + \varphi_2(z_2)] = T\sin\beta + A, \\ &2Re[\mu_1\varphi_1(z_1) + \mu_2\varphi_2(z_2)] = -T\cos\beta + B, \\ &z_1 \in L_1, z_2 \in L_2, \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} &2Re[\varphi_1(z_1) + \varphi_2(z_2)] = A', \\ &2Re[\mu_1\varphi_1(z_1) + \mu_2\varphi_2(z_2)] = B', \\ &z_1 \in L_1', z_2 \in L_2', \end{aligned}\right\} \tag{6}$$

where $A$, $B$, $A'$ and $B'$ are real constants to be determined. We note that these constants do not influence the final stress distribution in the half-plane. We mention also that the boundary condition (5) is obtained via integration of Eq. (2) with respect to the arc length $s$ along the curve $L$. In what follows, we will determine the complex potentials $\varphi_1(z_1)$ and $\varphi_2(z_2)$ in the entire regions $S_1$ and $S_2$, respectively, using the boundary conditions (5) and (6).

## 3. Solution procedure

Note that the domain of definition $S_i$ $(i = 1, 2)$ of the complex potential $\varphi_i(z_i)$ $(i = 1, 2)$ can be interpreted as the intersection of the infinite region (of the $z_i$-plane $(i = 1, 2)$) lying outside the hole bounded by $L_i$ $(i = 1, 2)$ with the entire lower $z_i$-half-plane (with no hole), so that, based on the principle of superposition, $\varphi_i(z_i)$ $(i = 1, 2)$ can be expressed in the form

$$\varphi_i(z_i) = \sum_{j=1}^{+\infty} a_{i,j}\,\xi_i^{-j} + \sum_{j=1}^{+\infty} b_{i,j}\,\eta_i^{j}, \quad i = 1, 2; \tag{7}$$

where $a_{i,j}$ and $b_{i,j}$ are constant coefficients to be determined. The $\xi_i$ – and $\eta_i$ – planes $(i = 1, 2)$ are associated with the $z_i$-plane $(i = 1, 2)$ via the conformal mappings [21,22],

$$\begin{aligned} &z_i = \omega_i(\xi_i) = \mu_i y_0 + \frac{a_0 - I\mu_i b_0}{2}\xi_i + \frac{\bar{a}_0 + I\mu_i\bar{b}_0}{2}\xi_i^{-1}, \\ &a_0 = a\cos\alpha + Ib\sin\alpha, \quad b_0 = b\cos\alpha + Ia\sin\alpha, \\ &|\xi_i| \geq 1, \quad i = 1, 2; \end{aligned} \tag{8}$$