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The anti-plane solution for the edge cracks originating from an arbitrary hole in a piezoelectric material

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ABSTRACT

In this paper, a general and simple way was found to solve the problem of an arbitrary hole with edge cracks in transversely isotropic piezoelectric materials based on the complex variable method and the method of numerical conformal mapping. Firstly, the approximate mapping function which maps the outside of the arbitrary hole and the cracks into the outside of a circular hole is derived after a series of conformal mapping process. Secondly, based on the assumption that the surface of the cracks and hole is electrically impermeable and traction-free, the approximate expressions for the complex potential, fields intensity factors and energy release rates are presented, respectively. Thirdly, under the in-plane electric loading together with the out-plane mechanical loading, the influences of the hole size, crack length and mechanical/electric loading on the fields intensity factors and energy release rates are analyzed. Finally, some particular holes with edge cracks are studied in numerical analysis. The result shows that, the mechanical loading always promotes crack growth, while the electric loading may retard crack growth.

1. Introduction

Structures with holes in piezoelectric materials have been widely used in practical engineering. Under the complicated loading environment, this structure can produce the so-called “stress concentration” phenomenon, which leads to the damage of material. Therefore, it is important to study the cracked hole problem in piezoelectric materials.

Many literature can be found for the fracture problems in piezoelectric materials, such as: McMeeking [1], Zhang and Gao [2], Schneider [3], Suo et al. [4], and Li et al. [5]. However, only a little literature focused on the cracked hole problems. Wang and Gao [6] modified the mapping function in Bowie [7] and then studied the mode III fracture problem of edge cracks originating from a circular hole in an infinite piezoelectric solid. Guo et al. [8,9] obtained the exact solutions for anti-plane problem of two asymmetrical edge cracks emanating from an elliptical hole in a piezoelectric material under the electrically impermeable and permeable assumption. By using the complex variable method, Lu [10] obtained the analytical expression for the complex potential in a half plane. Hasebe et al. [24] studied the oblique crack at the edge of a half plane, and the analytical solution was presented. However, those literatures are only concerning about a certain shape hole, e.g.: circular hole, elliptical hole and half plane. The solution of an arbitrary cracked hole remains unclear.

Gao and Noda [11] studied the anti-plane problem of an infinite piezoelectric material contains an arbitrary hole by using the Faber series and the complex variable method. According to Gao and Noda’s method, the mapping function is the key to solve the cracked hole problem. In order to solve the arbitrary cracked hole problem, the method of numerical conformal mapping is used to obtain the approximate mapping function in this paper. Although, the method of numerical conformal mapping is not a new topic in mathematics, and can be found in many literatures [12–15], but the method used in this paper is more direct and simple.

In this paper, some basic equations are given in Section 2, and detailed information of numerical conformal mapping function can be seen in Section 3. The complex potentials, fields intensity factors and energy release rate based on the impermeable and traction-free boundary conditions are derived in Section 4. In Section 5, two particular holes are studied to investigate the influence of the shape on the fields intensity factors and the energy release rate. Finally, main conclusions are summarized in Section 6.

2. Basic equations

In this paper, a transversely isotropic piezoelectric solid with the poling direction along the positive axis x_3 and the isotropic plane in the x_1–x_2 plane is considered. For the anti-plane problem, all the physical parameters depend on only two coordinate parameters

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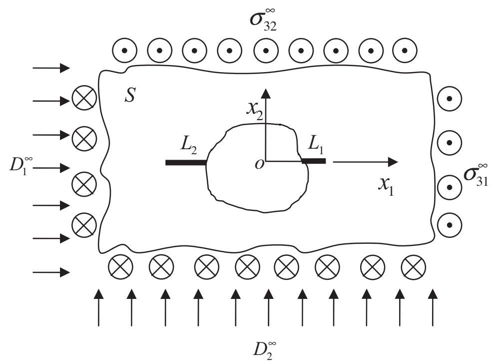

Fig. 1. Cracks at the edge of an arbitrary hole in an infinite piezoelectric solid.

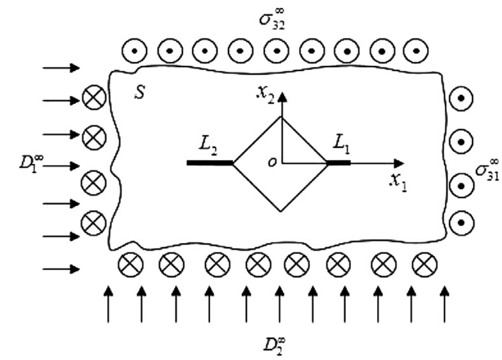

Fig. 2. Cracks at the edge of a square hole in an infinite piezoelectric solid.

(x_1, x_2). The mechanical loads perpendicular to the plane, while the electric loads parallel to the plane. Under this condition, one has [2,6,16]

$$\sigma_{31} = c_{44}\frac{\partial u_3}{\partial x_1} + e_{15}\frac{\partial \phi}{\partial x_1}, \quad \sigma_{32} = c_{44}\frac{\partial u_3}{\partial x_2} + e_{15}\frac{\partial \phi}{\partial x_2},$$
$$D_1 = e_{15}\frac{\partial u_3}{\partial x_1} - \varepsilon_{11}\frac{\partial \phi}{\partial x_1}, \quad D_2 = e_{15}\frac{\partial u_3}{\partial x_2} - \varepsilon_{11}\frac{\partial \phi}{\partial x_2}. \tag{1}$$

where, the volume force and charge density are ignored; σ_{ij}, D_i, u_i and ϕ stand for the stress, electric displacement, displacement and the potential, respectively; c_{ij}, e_{ij} and ε_{ij} are the elastic stiffness tensor, the piezoelectric coupling tensors and the dielectric permittivities, respectively.

The general solution of Eq. (1) can be expressed by the generalized stress function $\boldsymbol{\varphi}$ and generalized displacement $\mathbf{u}$, which is [2]

$$\mathbf{u} = (u_3, \phi)^T = \mathbf{A}\mathbf{f}(z) + \bar{\mathbf{A}}\overline{\mathbf{f}(z)}, \tag{2}$$

$$\boldsymbol{\varphi} = \mathbf{B}\mathbf{f}(z) + \bar{\mathbf{B}}\overline{\mathbf{f}(z)}, \quad (z = x_1 + ix_2), \tag{3}$$

where, $\mathbf{f}(z)$ is an unknown complex vector; $\mathbf{A}$ and $\mathbf{B}$ stand for the material constant matrices, defined as

$$\mathbf{A} = \mathbf{I},$$
$$\mathbf{B} = i\mathbf{B}_0 = i\begin{bmatrix} c_{44} & e_{15} \\ e_{15} & -\varepsilon_{11} \end{bmatrix}. \tag{4}$$

Standardized the matrixes $\mathbf{A}$ and $\mathbf{B}$, and the new matrixes $\mathbf{A}$ and $\mathbf{B}$ have the following relation [17]:

$$\begin{bmatrix} \mathbf{B}^T & \mathbf{A}^T \\ \bar{\mathbf{B}}^T & \bar{\mathbf{A}}^T \end{bmatrix} \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}. \tag{5}$$

Once the complex potential $\mathbf{f}(z)$ is obtained based on the given boundary conditions, all the fields can be determined from

$$(\sigma_{31}, D_1)^T = -\boldsymbol{\varphi}_{,2}, \quad (\sigma_{32}, D_2)^T = \boldsymbol{\varphi}_{,1},$$
$$E_1 = -u_{3,1}, \quad E_2 = -u_{3,2}. \tag{6}$$

where the comma stands for the partial differential, and E_i stands for the electric field.

3. Mapping function

An infinite piezoelectric solid contains an arbitrary cracked hole is shown in Fig. 1.

According to the Riemann theorem [18], there must be an analytic function which can maps an arbitrary simply connected domain into a circular hole. However, one cannot obtain the specific conformal mapping function by using this theorem. Therefore, we use a numerical method to obtain the approximate conformal mapping function here.

The Laurent series form extended at the infinite point of the mapping function $z = \omega(\zeta)$ can be expressed as

$$z = \omega(\zeta) = b_1\zeta + b_0 + b_{-1}\zeta^{-1} + b_{-2}\zeta^{-2} + \cdots, \tag{7}$$

where, $b_m = \frac{1}{2\pi i}\oint \frac{\omega(\zeta)}{\zeta^{m+1}}d\zeta \quad (m < 2)$.

Eq. (7) maps the outside of the cracks and hole into the outside of a circular hole. Let $\zeta = \zeta^{-1}$, and a new mapping function which maps the outside of the cracks and hole into the inside of a circular hole is obtained. If the tips of the cracked hole lie on the x_1 axis, the two tips correspond to ± 1 in the ζ-plane.

For an arbitrary cracked hole, the coefficients of Eq. (7) are unknown and not easy to be derived. However, these coefficients can be determined much easier for a hole without cracks. Some particular mapping functions can be found in Savin's research [19], which can be used as the baseline mapping functions. After a series of conformal mapping process, the mapping function which maps the outside of the arbitrary hole and cracks into the outside of a circular hole are obtained. This solution avoids the complicated process of numerical mapping method from the original problem.

Taking a square hole with two cracks as an example, as shown in Fig. 2. The mapping function which maps the outside of the square into the outside of a circular hole has the form as [19]:

$$z = \omega(\zeta) = R\left(\zeta - \frac{1}{6}\zeta^{-3} + \frac{1}{56}\zeta^{-7} - \frac{1}{176}\zeta^{-11} + \cdots\right), \tag{8}$$

where, R is a constant connected with the square length a, and $R = 0.5914a$ according to the correspondence of the points. Eq. (8) is an approximate mapping function, the square has a radius of curvature at the corner, which is $r = 0.014a$. The accuracy of the result can be controlled by the items of Eq. (8), and in this paper, the first four items are selected.

Fig. 3 shows the brief process of the conformal mapping. Based on Eq. (8), the mapping function which maps the outside of the cracks and square hole into the outside of a circular hole can be expressed as

$$z = \omega(\zeta) = R\left(\mu(\zeta^{-1}) + \frac{1}{6}\mu(\zeta^{-1})^{-3} + \frac{1}{56}\mu(\zeta^{-1})^{-7} + \frac{1}{176}\mu(\zeta^{-1})^{-11}\right), \tag{9}$$

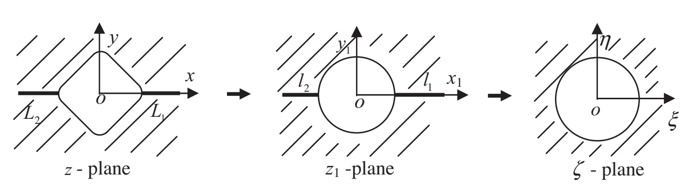

Fig. 3. Brief process of the conformal mapping of a square hole and cracks.