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Three-dimensional analytical elasticity solution for loaded functionally graded coated half-space

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ABSTRACT

In the present paper, a three-dimensional problem of elasticity of normal and tangential loading of surface of the functionally graded coated half-space is considered. In case when Poisson's ratio is constant and the Young's modulus is a power or exponential function of the distance from the surface of the half-space, analytical solution using Fourier transform is obtained. Stress field due to Hertz contact pressure in an elliptical region are studied as a function of the parameter b/a (where a and b are axes of the contact ellipse) and coating thickness.

1. Introduction

In the last decade the analysis of the stress in the inhomogeneous coating caused by contact pressure was subject of research by many scientists. Problems of the theory of elasticity for an inhomogeneous medium are described by a system of partial differential equations with variable coefficients.

Analytical solution of this equation is known only in few cases. Mainly the researcher consider the problem in which the Poissons ratio is constant and change of the Young's modulus (or shear modulus) is described by exponential function. Two-dimensional contact problems was analysed in the work [1–5]. Using Fourier integral transform method the analytical dependences between displacement components and functions described applied loading was obtained. Contact problem is reduced to a set of singular integral equations. Analytical method was using also to solve two-dimensional problem of thermoelasticity [6–8]. In these works the thermal conductivity coefficient, shear modulus and the thermal expansion coefficient vary exponentially in the thickness direction. The two-dimensional frictionless contact problem of a functionally graded magneto-electro-elastic materials layered half-plane under a rigid flat or a cylindrical punch was considered in [9]. Solution of axis-symmetrically problem of elasticity was obtained in [10–13]. The dependence of Young's modulus on the distance from the surface of the half-space is described also by an power function. The analytical solution of problem for nonhomogeneous half-space was obtained in [14–16]. It contains the modified Bessel functions. Similar problems for coating (or layer) was solved in [17–19]. Analytical methods is seldom used to solve the three-dimensional problems.

Parallel with the analytical methods of solution, it is also customary to use an approach according to which the analysed inhomogeneous coating is modelled by a stack of homogeneous or inhomogeneous layers [19–25]. To verify the obtained approximate solutions the analytical solutions was used.

In present paper we study the three-dimensional problem of elasticity for functionally graded coated half-space under normal and tangential traction. Using the Fourier integral transforms, the boundary value problems for partial differential equations is reduced to solution ordinary differential equations with variable coefficients. The displacement vector and stress tensor components was obtained in integral form, which integrands contain solution of this equations. In the case, when Poisson's ratio is constant and the Young's modulus is a power $E(z)=E_0(c+z)^\alpha$ or exponential $E(z)=E_0\exp(\alpha z)$ function of the distance from the surface of the half-space, analytical solution of the ordinary differential equations with variable coefficients was write in elementary or special function form. The distribution of the stresses in the nonhomogeneous coating was compare with corresponding stress distribution in the homogeneous coating [26].

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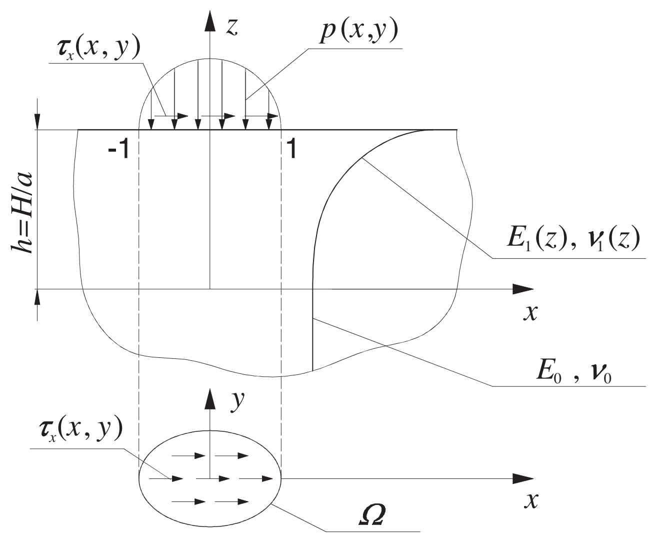

Fig. 1. The scheme of the problem of the elasticity for a functionally graded coated half-space.

2. Statement of the problem

To investigate the problem of elasticity we assume a dimensionless (related to the parameter a) coordinate system (x, y, z) in which the non-homogeneous half-space occupies the region $-\infty < x,y < \infty$ and $-\infty < z \leq h$ (Fig. 1). Considered half-space is composed of bonded together: homogeneous base and functionally graded coating ($0 \leq z < h$, $h = H/a$, H is the thickness of coating). The non-homogeneous material of coating characterized by Young's modulus $E_1(z)$ and Poisson ratio $\nu_1(z)$, where $E_1(z) \in C^1((0,h))$, $\nu_1(z) \in C^1((0,h))$. E_0 and ν_0 are the mechanical properties of material of homogeneous half-space. An arbitrary normal $p(x,y)$ and tangential $\tau_x(x,y)$ load are applied in the region Ω on the surface $z = h$. Assume that the conditions of perfect mechanical contact are realized between coating and substrate.

The problem is described by the partial differential equations

$$\Delta u_x^{(i)} + d_i \left(\frac{\partial \theta_1^{(i)}}{\partial x} + \frac{\partial^2 u_z^{(i)}}{\partial x \partial z} \right) + \breve{\mu}_i \left(\frac{\partial u_x^{(i)}}{\partial z} + \frac{\partial u_z^{(i)}}{\partial x} \right) = 0, \quad i = 0, 1, \tag{1}$$

$$\Delta u_y^{(i)} + d_i \left(\frac{\partial \theta_1^{(i)}}{\partial y} + \frac{\partial^2 u_z^{(i)}}{\partial y \partial z} \right) + \breve{\mu}_i \left(\frac{\partial u_y^{(i)}}{\partial z} + \frac{\partial u_z^{(i)}}{\partial y} \right) = 0, \quad i = 0, 1, \tag{2}$$

$$\Delta u_z^{(i)} + d_i \left(\frac{\partial \theta_1^{(i)}}{\partial z} + \frac{\partial^2 u_z^{(i)}}{\partial z^2} \right) + \left(\breve{\lambda}_i \theta_1^{(i)} + (\breve{\lambda}_i + 2\breve{\mu}_i) \frac{\partial u_z^{(i)}}{\partial z} \right) = 0, \quad i = 0, 1, \tag{3}$$

and the boundary conditions

$$\sigma_{xz}^{(1)}(x, y, z = h) = \tau_x(x, y) H(x, y), \tag{4a}$$

$$\sigma_{yz}^{(1)}(x, y, z = h) = 0, \tag{4b}$$

$$\sigma_{zz}^{(1)}(x, y, z = h) = -p(x, y) H(x, y), \tag{4c}$$

$$u^{(1)}(x, y, z = 0) = u^{(0)}(x, y, z = 0), \tag{5}$$

$$\sigma^{(1)}(x, y, z = 0) n = \sigma^{(0)}(x, y, z = 0) n, \tag{6}$$

$$u^{(i)}(x, y, z) \to 0, \quad x^2 + y^2 + z^2 \to \infty, \tag{7}$$

where $\mathbf{u}^{(i)}$ is the vector of the non-dimensional displacement referred to the parameter a; $\boldsymbol{\sigma}^{(i)}$ is the stress tensor; the superscripts $i = 0$ and $i = 1$ correspond to the parameters and functions of state in the homogeneous half space and in the inhomogeneous layer, respectively;

$$\theta_1^{(i)} = \frac{\partial u_x^{(i)}}{\partial x} + \frac{\partial u_y^{(i)}}{\partial y}, \quad i = 0, 1; \tag{8}$$

$\breve{\mu}_0 = \breve{\lambda}_0 = 0$; $\breve{\mu}_1(z) = \mu_1'/\mu_1$; $\breve{\lambda}_1(z) = \lambda_1'/\mu_1$; λ_i, μ_i, $i = 0,1$ are Lamé constants; $\lambda_1'(z) = d\lambda_1/dz$, $\mu_1'(z) = d\mu_1/dz$, $d_i = 1/(1 - 2\nu_i)$; $H(x,y)$ is the Heaviside unit step function ($H(x,y) = 1$, where $(x,y) \in \Omega$ and $H(x,y) = 0$, where $(x,y) \notin \Omega$); $\boldsymbol{n} = (0, 0, 1)$.

3. The method of solution

Eqs. (1)–(3) may be solved by use of the Fourier transform technique:

$$\tilde{f}(\xi, \eta, z) = F\left(f(x, y, z), x \to \xi, y \to \eta\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) \exp\left(-ix\xi - iy\eta\right) dxdy. \tag{9}$$

We obtain differential equations:

$$\frac{d^2 \tilde{u}_x^{(i)}}{dz^2} - s^2 \tilde{u}_x^{(i)} + d_i i\xi \left(\tilde{\theta}_1^{(i)} + \frac{d\tilde{u}_z^{(i)}}{dz} \right) + \breve{\mu}_i \left(\frac{d\tilde{u}_x^{(i)}}{dz} + i\xi \tilde{u}_z^{(i)} \right) = 0, \quad i = 0, 1, \tag{10}$$

$$\frac{d^2 \tilde{u}_y^{(i)}}{dz^2} - s^2 \tilde{u}_y^{(i)} + d_i i\eta \left(\tilde{\theta}_1^{(i)} + \frac{d\tilde{u}_z^{(i)}}{dz} \right) + \breve{\mu}_i \left(\frac{d\tilde{u}_y^{(i)}}{dz} + i\eta \tilde{u}_z^{(i)} \right) = 0, \quad i = 0, 1, \tag{11}$$

$$\frac{d^2 \tilde{u}_z^{(i)}}{dz^2} - s^2 \tilde{u}_z^{(i)} + d_i \left(\frac{d\tilde{\theta}_1^{(i)}}{dz} + \frac{d^2 \tilde{u}_z^{(i)}}{dz^2} \right) + \left(\breve{\lambda}_i \tilde{\theta}_1^{(i)} + (\breve{\lambda}_i + 2\breve{\mu}_i) \frac{d\tilde{u}_z^{(i)}}{dz} \right) = 0, \quad i = 0, 1, \tag{12}$$

Where, $s^2 = \xi^2 + \eta^2$.

Calculating $i\eta$(Eq. (10)) − $i\xi$(Eq. (11)) and $i\xi$(Eq. (10)) + $i\eta$(Eq. (11)), we obtain the following equations:

$$\frac{d^2 \tilde{\chi}^{(i)}}{dz^2} + \breve{\mu}_i \frac{d\tilde{\chi}^{(i)}}{dz} - s^2 \tilde{\chi}^{(i)} = 0, \quad i = 0, 1, \tag{13}$$

$$\frac{d^2 \tilde{\theta}_1^{(i)}}{dz^2} + \breve{\mu}_i \frac{d\tilde{\theta}_1^{(i)}}{dz} - s^2 (1 + d_i) \tilde{\theta}_1^{(i)} - d_i s^2 \frac{d\tilde{u}_z^{(i)}}{dz} - \breve{\mu}_i s^2 \tilde{u}_z^{(i)} = 0, \quad i = 0, 1, \tag{14}$$

Where,

$$\tilde{\chi}^{(i)} = i\eta \tilde{u}_x^{(i)} - i\xi \tilde{u}_y^{(i)}, \quad \chi^{(i)} = \frac{\partial u_x^{(i)}}{\partial y} - \frac{\partial u_y^{(i)}}{\partial x}, \quad i = 0, 1. \tag{15}$$

The general solution of Eqs. (12), (13) and (14) in transform domain, satisfied conditions (7) can be written in the form:

$$\tilde{\chi}^{(0)}\left(\xi, \eta, z\right) = b_0\left(\xi, \eta\right) \exp(sz), \tag{16}$$

$$\tilde{\chi}^{(1)}\left(\xi, \eta, z\right) = b_1\left(\xi, \eta\right) \chi_1(s, z) + b_2\left(\xi, \eta\right) \chi_2(s, z), \tag{17}$$