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An iterative approach for nonproportionally damped systems

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ABSTRACT

Modal analysis of nonproportionally damped linear dynamic systems is considered. Dynamic response of such systems can be expressed by a modal series in terms of complex modes. Normally state-space based methods or approximate perturbation methods are necessary for the computation of complex modes. In this paper, an iterative method to calculate complex modes from classical normal modes for general linear systems is proposed. A simple numerical algorithm is developed to implement the iterative method. The new method is illustrated using a numerical example.

1. Introduction

The equation of motion of an n-degree-of-freedom linear viscously damped system can be expressed by coupled differential equations as

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t). \tag{1}$$

Here $\mathbf{u}(t)\in\mathbb{R}^n$ is the displacement vector, $\mathbf{f}(t)\in\mathbb{R}^n$ is the forcing vector, $\mathbf{M},\mathbf{K},\mathbf{C}\in\mathbb{R}^{n\times n}$ are respectively the mass matrix, stiffness and the viscous damping matrix. In general $\mathbf{M}$ is a positive definite symmetric matrix, $\mathbf{C}$ and $\mathbf{K}$ are non-negative definite symmetric matrices. The natural frequencies ($\omega_j\in\mathbb{R}$) and the mode shapes ($\mathbf{x}_j\in\mathbb{R}^n$) of the corresponding undamped system can be obtained (Meirovitch, 1997) by solving the matrix eigenvalue problem

$$\mathbf{K}\mathbf{x}_j=\omega_j^2\mathbf{M}\mathbf{x}_j,\quad \forall j=1,2,\ldots,n. \tag{2}$$

The undamped eigenvectors satisfy an orthogonality relationship over the mass and stiffness matrices, that is

$$\mathbf{x}_k^T\mathbf{M}\mathbf{x}_j=\delta_{kj} \tag{3}$$

and

$$\mathbf{x}_k^T\mathbf{K}\mathbf{x}_j=\omega_j^2\delta_{kj},\forall k,\quad j=1,2,\ldots,n \tag{4}$$

where δ_{kj} is the Kroneker delta function. We construct the modal matrix

$$\mathbf{X}=[\mathbf{x}_1,\mathbf{x}_2,\quad\ldots,\mathbf{x}_n]\in\mathbb{R}^n. \tag{5}$$

The modal matrix can be used to diagonalize system (1) provided the damping matrix $\mathbf{C}$ is simultaneously diagonalizable with $\mathbf{M}$ and $\mathbf{K}$. This condition, known as the proportional damping, originally introduced by Lord Rayleigh (Rayleigh, 1877) in 1877, is still in wide use today. The mathematical condition for proportional damping can be obtained from the commutative behaviour of the system matrices (Caughey and O'Kelly, 1965). This can be expressed as $\mathbf{C}\mathbf{M}^{-1}\mathbf{K}=\mathbf{K}\mathbf{M}^{-1}\mathbf{C}$ or equivalently $\mathbf{C}=\mathbf{M}f(\mathbf{M}^{-1}\mathbf{K})$ as shown by Adhikari (2006). The concern of this paper is when this condition is not met, the most likely case for many practical applications. In particular, due to the recent developments in actively controlled structures and the increasing use of composite and smart materials, the need to consider general nonproportionally damped linear dynamic systems is more than ever before.

For nonproportionally damped systems, the modal damping matrix

$$\mathbf{C}'=\mathbf{X}^T\mathbf{C}\mathbf{X} \tag{6}$$

is not a diagonal matrix. Such problems can be solved using a spectral approach similar to the undamped or proportionally damped system by transforming Eq. (1) into a state-space form (Meirovitch, 1997). The state-space approach is not only computationally more expensive, it also lacks the physical insight provided by the classical normal mode based approach. Therefore, many authors have developed approximate methods in the original space. Rayleigh (1877) proposed a perturbation method which form the basis of many contemporary approximation methods (Adhikari, 1999a; ElBeheiry, 2009; Adhikari, 1999b). It is now known that either the frequency separation between the normal modes (Hasselsman, 1976), often known as 'Hasselsman's criteria', or some form of diagonal dominance (Shahruz and Ma, 1988; Morzfeld et al., 2009; Adhikari, 2004; Morzfeld et al., 2008) in the modal damping matrix $\mathbf{C}'$ is sufficient for neglecting modal coupling. In a recent work, Udwadia (2009) proved that for systems with non-repeated eigenvalues, the best

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approximation of a diagonal modal damping matrix is simply to consider the diagonal of the $\mathbf{C}'$ matrix.

The eigenvalue problem corresponding to system (1) can be expressed as

$$[s_j^2\mathbf{M} + s_j\mathbf{C} + \mathbf{K}]\mathbf{u}_j = 0, \quad \forall j = 1, 2, \ldots, 2n \tag{7}$$

where $s_j \in \mathbf{C}$ are the eigenvalues and $\mathbf{u}_j \in \mathbf{C}^n$ are the eigenvectors. Comprehensive details on this type of quadratic eigenvalue problem can be found in Tisseur and Meerbergen (2001). Since **M**, **K** and **C** are all real matrices, the eigensolutions are either real or they appear in the complex conjugate pairs. In this paper we consider complex conjugate eigensolutions only as for stable systems such eigenvalues are of great practical importance. Using the eigensolutions, the frequency response function (FRF) can be obtained (see for example (Adhikari, 1999a; Tisseur and Meerbergen, 2001)) as

$$\mathbf{H}(i\omega) = \sum_{j=1}^{n} \left[\frac{\gamma_j \mathbf{u}_j \mathbf{u}_j^T}{i\omega - s_j} + \frac{\gamma_j^* \mathbf{u}_j^* \mathbf{u}_j^{*T}}{i\omega - s_j^*} \right] \quad \text{where}$$

$$\gamma_j = \frac{1}{\mathbf{u}_j^T [2s_j\mathbf{M} + \mathbf{C}]\mathbf{u}_j}. \tag{8}$$

Here $(\bullet)^*$ denotes complex conjugation and $(\bullet)^T$ denotes matrix transposition. This equation shows that if the complex eigensolutions s_j and $\mathbf{u}_j$ can be obtained efficiently, the dynamic response can be obtained exactly using Eq. (8). In this paper an iterative approach is developed to obtain the complex eigensolutions of nonproportionally damped systems from the undamped eigensolutions.

2. Iterative approach for the eigensolutions

Ibrahimbegovic and Wilson (1989) have developed a procedure for analyzing non-proportionally damped systems using a subspace with a vector basis generated from the mass and stiffness matrices. Their approach avoids the use of complex eigensolutions. In the time domain, an iterative approach for solving the coupled equations was developed by Udwadia and Esfandiari (1990) based on updating the forcing term appropriately. In the method proposed here, we obtain the complex modes and complex frequencies in an iterative manner.

For distinct undamped eigenvalues (ω_l^2), $\mathbf{x}_l$, $\forall l = 1, \ldots, n$, form a complete set of vectors. For this reason, $\mathbf{u}_j$ can be expanded as a complex linear combination of $\mathbf{x}_l$. Thus, an expansion of the form

$$\mathbf{u}_j = \sum_{l=1}^{n} \alpha_l^{(j)} \mathbf{x}_l \tag{9}$$

may be considered. Without any loss of generality, we can assume that $\alpha_j^{(j)} = 1$ (normalization) which leaves us to determine $\alpha_l^{(j)}, \forall l \neq j$. Substituting the expansion of $\mathbf{u}_j$ into the eigenvalue equation (7), one obtains the approximation error for the j-th mode as

$$\varepsilon_j = \sum_{l=1}^{n} s_j^2 \alpha_l^{(j)} \mathbf{M}\mathbf{x}_l + s_j \alpha_l^{(j)} \mathbf{C}\mathbf{x}_l + \alpha_l^{(j)} \mathbf{K}\mathbf{x}_l. \tag{10}$$

We use a Galerkin approach to minimize this error by viewing the expansion (9) as a projection in the basis functions $x_l \in \mathbb{R}^n, \quad \forall l = 1, 2, \ldots, n$. Therefore, we make the error orthogonal to the basis functions, that is

$$\varepsilon_j \perp \mathbf{x}_l \quad \text{or} \quad x_k^T \varepsilon_j = 0, \quad \forall k = 1, 2, \ldots, n. \tag{11}$$

Using the orthogonality property of the undamped eigenvectors described by (3) and (4) one obtains

$$s_j^2 \alpha_k^{(j)} + s_j \sum_{l=1}^{n} \alpha_l^{(j)} C'_{kl} + \omega_k^2 \alpha_k^{(j)} = 0, \quad \forall k = 1, \ldots, n \tag{12}$$

where $C'_{kl} = \mathbf{x}_k^T \mathbf{C} \mathbf{x}_l$ are the elements of the modal damping matrix $\mathbf{C}'$ defined in Eq. (6). The j-th equation of this set obtained by setting $k = j$ can be written as

$$(s_j^2 + s_j C'_{jj} + \omega_j^2)\alpha_j^{(j)} + s_j \sum_{l \neq j}^{n} \alpha_l^{(j)} C'_{jl} = 0. \tag{13}$$

Recalling that $\alpha_j^{(j)} = 1$ and $\mathbf{C}'$ is a symmetric matrix, this equation can be rewritten as

$$s_j^2 + s_j \underbrace{\left(C'_{jj} + \sum_{l \neq j}^{n} \alpha_l^{(j)} C'_{lj} \right)}_{\gamma_j} + \omega_j^2 = 0 \tag{14}$$

where

$$\gamma_j = C'_{jj} + \mathbf{b}_j^T \mathbf{a}_j \tag{15}$$

$$\mathbf{b}_j = \{C'_{1j}, C'_{2j}, \ldots, \{j\text{-th term deleted}\}, \ldots, C'_{nj}\}^T \in \mathbb{R}^{(n-1)} \tag{16}$$

and

$$\mathbf{a}_j = \{\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \{j\text{-th term deleted}\}, \ldots, \alpha_n^{(j)}\}^T \in \mathbb{C}^{(n-1)} \tag{17}$$

The vector $\mathbf{a}_j$ is unknown and can be obtained by excluding the $j = k$ case in Eq. (12). Excluding this case one has

$$s_j^2 \alpha_k^{(j)} + s_j \left(C'_{kj} + \alpha_k^{(j)} C'_{kk} + \sum_{l \neq k \neq j}^{n} \alpha_l^{(j)} C'_{kl} \right) + \omega_k^2 \alpha_k^{(j)} = 0, \quad \text{or}$$

$$(s_j^2 + \omega_k^2 + C'_{kk})\alpha_k^{(j)} + s_j \sum_{l \neq k \neq j}^{n} C'_{kl} \alpha_l^{(j)} = -s_j C'_{kj}, \quad \forall k = 1, \ldots, n; \neq j. \tag{18}$$

These equations can be combined into a matrix form as

$$[\mathbf{P}_j - \mathbf{Q}_j]\mathbf{a}_j = \mathbf{b}_j. \tag{19}$$

In the above equation, the vectors $\mathbf{a}_j$ and $\mathbf{b}_j$ have been defined before. The matrices $\mathbf{P}_j$ and $\mathbf{Q}_j$ are defined as

$$\mathbf{P}_j = \text{diag} \left[\frac{s_j^2 + s_j C'_{11} + \omega_1^2}{-s_j}, \ldots, \{j\text{-th term deleted}\}, \ldots, \right.$$
$$\left. \times \frac{s_j^2 + s_j C'_{NN} + \omega_n^2}{-s_j} \right] \in \mathbb{C}^{(n-1)\times(n-1)}, \tag{20}$$

and

$$\mathbf{Q}_j = \begin{bmatrix} 0 & C'_{12} & \ldots & \{j\text{-th term deleted}\} & \ldots & C'_{1n} \\ C'_{21} & 0 & \vdots & \vdots & \vdots & C'_{2n} \\ \vdots & \vdots & \vdots & \{j\text{-th term deleted}\} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C'_{n1} & C'_{n2} & \ldots & \{j\text{-th term deleted}\} & \ldots & 0 \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}. \tag{21}$$

From Eq. (19), $\mathbf{a}_j$ should be obtained by solving the set of linear equations. Because $\mathbf{P}_j$ is a diagonal matrix, one way to do this is