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Perturbation solution for the 2D Boussinesq equation

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ABSTRACT

Boussinesq equation arises in shallow water flows and in elasticity of rods and shells. It contains nonlinearity and fourth-order dispersion and has been one of the main soliton models in 1D. To find its 2D solutions, a perturbation series with respect to the small parameter $\varepsilon = c^2$ is developed in the present work, where c is the phase speed of the localized wave. Within the order $O(\varepsilon^2) = O(c^4)$, a hierarchy is derived consisting of one-dimensional fourth-order equations. The Bessel operators involved are reformulated to facilitate the creation of difference schemes for the ODEs from the hierarchy. The numerical scheme uses a special approximation for the behavioral condition in the singularity point (the origin). The results of this work show that at infinity the stationary 2D wave shape decays algebraically, rather than exponentially as in the 1D cases. The new result can be instrumental for understanding the interaction of 2D Boussinesq solitons, and for creating more efficient numerical algorithms explicitly acknowledging the asymptotic behavior of the solution.

1. Introduction

Boussinesq's equation (BE) was the first model for the propagation of surface waves over shallow inviscid fluid layer. Boussinesq (1871, 1872) developed a perturbation method to solve the Laplace equation in the bulk, and to consequently close the system that contains only the surface variable. He arrived at a generalized wave equation (GWE) that contains dispersion in addition to the standard terms. For a slowly evolving wave in a coordinate frame moving with the center of the wave, BE reduces to the Korteweg–de Vries equation which is widely studied in 1D. The approach developed by Boussinesq opened a new avenue of modeling: the 'amplitude equations'. He found an analytical solution of his equation and thus proved that the balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the wave. This discovery can be properly termed 'Boussinesq paradigm'.

Apart from the significance for the shallow water flows, this paradigm is very important for understanding the particle-like behavior of nonlinear localized waves. In the 1960s it was discovered that the permanent waves can behave in many instances as particles (the so-called 'collision property'), and were called *solitons* by Zabusky and Kruskal (1965). The localized waves which can retain their identity during interaction appear to be a rather pertinent model for particles, especially if some mechanical properties (such as mass, energy, and momentum) are conserved by the governing system of equations. In 1D, a plethora of deep mathematical results have been obtained for solitons (see Ablowitz and Segur, 1981; Newell, 1985). The success was contingent upon the existence of an analytical solution of the respective nonlinear dispersive equation. As it should have been expected, most of the physical systems are not fully integrable (even in one spatial dimension) and only a numerical approach can lead to unearthing the pertinent physical mechanisms of the interactions (see, e.g., Christov and Velarde, 1994; Christov, 2001 and the literature cited therein).

The overwhelming majority of the analytical and numerical results obtained so far are for one spatial dimension, while in multidimension, much less is possible to achieve analytically, and almost nothing is known about the unsteady solutions that involve interactions, especially when the full-fledged Boussinesq equations are involved. The 2D case is relatively better studied for the so-called Kadomtsev–Petviashvili equation (KPE), which has fourth derivatives only in one of the spatial directions, while it is second-order in the other direction. Interesting analytical results are obtained for the solutions of KPE, which are localized in the direction with the fourth-order derivative, and are periodic in the other direction (see, e.g., Christov et al., 2007; Porubov et al., 2004,? and the literature cited therein).

For the time being, the 2D Boussinesq model is still less accessible analytically, which requires developing numerical techniques. The first case to undergo investigation is the steadily propagating wave profile. Some preliminary numerical results were obtained by Choudhury and Christov (2005). One of the main difficulties for the difference schemes lies in the inevitable reducing of the infinite interval to a finite one. This can be surmounted if a spectral method

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is used with a basis system of localized functions which automatically acknowledge the requirement that the solution belongs to $L^2(-\infty, \infty)$ space. Along these lines, a specialized Galerkin spectral technique was propsed in Christov (1982), and applied to various 1D problems. Christov (1995b) created a spectral scheme for a 2D problem related to the quadratic Klein–Gordon equation (KGE). The application of the Galerkin procedure to Boussinesq model can be found in Christou and Christov (2007, 2009) for the stationary propagating 2D Boussinesq wave.

In the present work, we undertake an asymptotic semi-analytical solution for moderate phase speeds and compare the results with the above mentioned numerical works.

2. Boussinesq paradigm equation (BPE)

Following Boussinesq (1872), we restrict the derivations to the case when the shape function $h(x, y, t)$ of the free surface is single-valued, i.e., there is no wave breaking. The motion in the bulk is governed by the Laplace equation for the potential Φ. We introduce dimensionless variables according to the scheme $\Phi = UL\phi$, $h = H\eta$, $z = Hz'$, $x = Lx'$, $y = Ly'$, $t = LU^{-1}t'$, where H is the scale for the vertical spatial coordinate (the thickness of the shallow layer) and L is the wave length in the horizontal plane. Respectively, $U = \sqrt{gH}$ is the characteristic scale for the velocity. Henceforth, the primes will be omitted.

As shown in Christov (2001), the consistent implementation of the Boussinesq method yields the following generalized wave equation (GWE) for $f = \phi(x, y, 0; t)$:

$$f_{tt} + 2\beta\nabla f \cdot \nabla f_t + \beta f_t \Delta f + \frac{3\beta^2}{2}(\nabla f)^2 \Delta f - \Delta f + \frac{\beta}{6}\Delta^2 f - \frac{\beta}{2}\frac{\partial^2 \Delta f}{\partial t^2} = 0. \tag{1}$$

Eq. (1) is the most rigorous amplitude equation that can be derived for the surface waves over an inviscid shallow layer, when the length of the wave is considered large in comparison with the depth of the layer. Since it was derived only in 2001, it has not attracted much attention, and the plethora of different inconsistent Boussinesq equations are still vigorously investigated. Each specific simplification of the general model reveals some particular trait of the balance between the nonlinearity and dispersion. In many cases, the resulting model that is integrable (see the original Boussinesq equation containing only fourth-order spatial derivatives for the dispersion). Such a feature is clearly important for advancing the specific soliton techniques. Much has been done to compare the behavior of the solutions of the different versions of Boussinesq equations. For instance, it was shown numerically in Christov and Velarde (1994) that the soliton interactions are qualitatively very similar for the non-integrable equation with mixed fourth derivative and the original integrable Boussinesq equation. This means that some aspects of the actual physics can be captured successfully by a specific version of the Boussinesq equation. The most popular are the versions that contain a quadratic nonlinearity, and we feature the new technique proposed here for this case.

Unfortunately, Boussinesq did some additional (and as it turns out) unnecessary assumptions, which rendered his equation incorrect in the sense of Hadamard. We term the original model the 'Boussinesq's Boussinesq equation' or BBE. During the years, it was 'improved' in a number of works. An overview of the different Boussinesq equations can be found in Christov and Velarde (1994), and the literature cited therein. The mere change of the incorrect sign of the fourth derivative in BBE yields the so-called 'good' or 'proper' Boussinesq equation, which we will refer to as the Boussinesq equation or BE. A different approach to removing the incorrectness of the BBE was discussed in Benjamin et al. (1972); Bogolubsky (1977); Manoranjan et al. (1988), and the situation was remedied by changing the spatial fourth derivative to a mixed fourth derivative, which resulted into an equation know nowadays as the regularized long wave equation (RLWE) or Benjamin–Bona–Mahony equation (BBME). In fact, the mixed derivative occurs naturally in Boussinesq derivation (see Eq. (1)), and was changed by Boussinesq to a fourth spatial derivative under an assumption that $\partial_t \approx c\,\partial_x$, which is currently known as the 'linear impedance relation' (or LIA). The LIA has gained quite a currency in different fields of fluid mechanics and has produced innumerable instances of unphysical results (see Christov et al., 2007 for the case in nonlinear acoustics).

Boussinesq applied the LIA also to the nonlinear terms, and neglected the cubic nonlinearity. This simplified the nonlinear terms of Eq. (1) to a point where Boussinesq was able to find the first *sech* solution for the permanent localized wave, proving thus the existence of the balance between the nonlinearity and dispersion. The actual nonlinearity is important because it provides for the Galilean invariance of the model (see Christov, 2001). Yet for the purposes of understanding the 2D solutions, one may find it useful to stay within the Boussinesq framework of simplifications, as far as the nonlinear terms are concerned. We focus here on the following two-dimensional amplitude equation:

$$w_{tt} = \Delta\left[w - \alpha w^2 + \beta_1 w_{tt} - \beta_2 \Delta w\right], \tag{2}$$

where w is the surface elevation, $\beta_1, \beta_2 > 0$ are two dispersion coefficients, and α is an amplitude parameter, which can be set equal to unity without loosing the generality. We term Eq. (2) the Boussinesq paradigm equation (or BPE). As already above mentioned, the main difference here is that BPE features one more term than BE, namely $\beta_1 \neq 0$. A note on the notation: in the original BE as related to the water waves, the nonlinear term has a positive sign, and the soltions are actually depressions for the subcritical case. Here we have deliberately changed the sign for the sake of the presentation.

It was shown in Christov (1995a, 2001) that the 1D PBE admits soliton solutions given by

$$w^s(x, t; c) = -\frac{3}{2\alpha}(c^2 - 1)\times\operatorname{sech}^2\left[\frac{1}{2}(x - ct)\sqrt{(c^2-1)/(\beta_1 c^2 - \beta_2)}\right], \tag{3}$$

where c is the phase speed. The soliton Eq. (3) is that it exists for $|c| > \max\{1, \sqrt{\beta_2/\beta_1}\}$ or $|c| < \min\{1, \sqrt{\beta_2/\beta_1}\}$. The first case is comprised by the so-called 'supercritical' solitons, while the latter encompasses the 'subritical' ones.

We set the amplitude parameter $\alpha = 1$, because it can always be eliminated by rescaling the solution. We can also select $\beta_2 = 1$. This leaves us with only one parameter, β_1, apart from the phase speed c.

For the numerical interaction of 2D Boussinesq solitons, one needs the shape of a stationary moving solitary wave in order to construct an initial condition. To this end, introduce relative coordinates $\hat{x} = x - c_1 t$, $\hat{y} = y - c_2 t$, in a frame moving with velocity (c_1, c_2). Since there is no evolution in the moving frame $v(x, y, t) = u(\hat{x}, \hat{y})$, and the following equation holds for u:

$$\begin{aligned}(c_1^2 u_{\hat{x}\hat{x}} + 2c_1c_2 u_{\hat{x}\hat{y}} + c_2^2 u_{\hat{y}\hat{y}}) = {} & (u_{\hat{x}\hat{x}} + u_{\hat{y}\hat{y}}) - \left[((u^2)_{\hat{x}\hat{x}} + (u^2)_{\hat{y}\hat{y}})\right] \\ & - (u_{\hat{x}\hat{x}\hat{x}\hat{x}} + 2u_{\hat{x}\hat{x}\hat{y}\hat{y}} + u_{\hat{y}\hat{y}\hat{y}\hat{y}}) \\ & + \beta_1\left[c_1^2(u_{\hat{x}\hat{x}\hat{x}\hat{x}} + u_{\hat{x}\hat{x}\hat{y}\hat{y}})\right. \\ & + 2c_1c_2(u_{\hat{x}\hat{x}\hat{x}\hat{y}} + u_{\hat{x}\hat{y}\hat{y}\hat{y}}) \\ & \left. + c_2^2(u_{\hat{x}\hat{x}\hat{y}\hat{y}} + u_{\hat{y}\hat{y}\hat{y}\hat{y}})\right]. \end{aligned} \tag{4}$$

The so-called asymptotic boundary conditions (a.b.c.) read $u \to 0$, for $\hat{x} \to \pm\infty$, $\hat{y} \to \pm\infty$. The a.b.c.'s are invariant under rotation of the coordinate system, hence it is enough to consider solitary propagating along one of the coordinate axes, only. We chose $c_1 = 0$,