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The role of the rotation matrix in the teaching of planar kinematics

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ABSTRACT

A prevalent tool in the teaching of planar kinematics is Euler's formula, namely, $e^{j\theta} = \cos\theta + j\sin\theta$, with j defined as $j^2 \equiv -1$, regarded as an operator in the complex-number plane. This operator maps every complex number $z = x + jy$ into $x\cos\theta - y\sin\theta + j(x\sin\theta + y\cos\theta)$. When z is represented as a vector in the complex plane, $e^{j\theta}z$ is z rotated through an angle θ ccw. This works and helps derive expressions for the position, velocity, and acceleration of a point of a rigid body in a systematic way. The downside of this practice is twofold: (i) it hides the geometrical meaning of the rotation; and (ii) it prevents the generalization of the operator to three dimensions. As an alternative to the use of Euler's formula, the 2×2 rotation matrix is available, but its use is not common in textbooks. The purpose of this paper is to stress the benefits of the use of this matrix in two dimensions, its straightforward generalization to three dimensions, and a possible extension to higher dimensions, if the need arises. An example of the use of the rotation matrix in planar kinematics is included, by means of an example taken from an application in mobile robotics.

1. Introduction

The use of the unit complex number $u(\theta) \equiv e^{j\theta}$, where j denotes the imaginary unit, in the teaching of planar kinematics is well established, as many textbooks have adopted it, e.g., [1] and [2]. Moreover, if $\theta = \theta(t)$, then

$$\dot{u} = \dot{\theta}\,(-\sin\theta + j\cos\theta), \quad \ddot{u} = \ddot{\theta}(-\sin\theta + j\cos\theta) - \dot{\theta}^2(\cos\theta + j\sin\theta). \tag{1}$$

Furthermore, $\ddot{u}$ can be expressed alternatively as

$$\ddot{u} = \frac{\ddot{\theta}}{\dot{\theta}}\dot{u} - \dot{\theta}^2 u \equiv \ddot{\theta}u'(\theta) - \dot{\theta}^2 u \tag{2}$$

Now, given that $\dot{u}$ can be expressed as

$$\dot{u} = \dot{\theta}\left[\cos\left(\theta + \frac{\pi}{2}\right) + j\sin\left(\theta + \frac{\pi}{2}\right)\right] \equiv \dot{\theta}\,u'(\theta).$$

the first expression of Eq. (1) shows that $\dot{u}$, in its vector representation, is normal to the vector representation of $u(t)$. The second expression of the same Eq. (1), rewritten in Eq. (2), shows a natural decomposition of $\ddot{u}$ into two orthogonal components, the first normal, the second parallel to $u(t)$, again, when the two complex numbers are represented as vectors.

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While $u(\theta)$ rightfully acts as an operator that maps the complex number $z = x + jy$ into a new complex number rotated through an angle θ ccw, in this author's opinion this operator hides the geometric and kinematic meaning of a rotation. Moreover, an extension to three dimensions is not apparent. As a matter of fact, Sir William Rowan Hamilton, best known for his discovery, with Olinde Rodrigues, of quaternions,[1] arrived at this concept while searching for an extension of complex numbers to three dimensions [3].

In the author's opinion, the introduction of complex numbers to represent points in the real plane is an artifice that can be avoided if the 2×2 rotation matrix $\mathbf{Q}_2$ is introduced, namely,

$$\mathbf{Q}_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{3}$$

The first objective of the paper is to shed light on the relation between $\mathbf{Q}_2$ and the complex number $u(\theta)$ introduced above.

2. The 2 × 2 Rotation matrix and the unit complex number

As pointed out by Bottema and Roth [4], $\mathbf{Q}_2$ can be expressed as

$$\mathbf{Q}_2 = \cos\theta 1_2 + \sin\theta \mathbf{E}_2, \quad \mathbf{E}_2 \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{4}$$

where 1_2 denotes the 2×2 identity matrix. Moreover, Eq. (4) reveals that $\mathbf{Q}_2$ is nothing but $e^{\theta \mathbf{E}_2}$. Indeed, if the Cayley–Hamilton Theorem [5] is recalled, $e^{\theta \mathbf{E}_2}$ can be expressed as a linear combination of $\mathbf{E}_2^0 = 1_2$ and $\mathbf{E}_2$, namely,

$$e^{\theta \mathbf{E}_2} = c_0 1_2 + \mathbf{c}_1 \theta \mathbf{E}_2 \tag{5}$$

with coefficients c_0 and c_1 as yet to be determined. These are obtained from the expansion of $f(\lambda_i) = e^{\lambda_i}$, where λ_i, for $i = 1, 2$, denotes the ith eigenvalue of $\theta \mathbf{E}_2$. The eigenvalues of interest can be readily found to be

$$\lambda_1 = j\theta, \quad \lambda_2 = -j\theta. \tag{6}$$

Now, coefficients c_0 and c_1 are obtained upon expressing e^{λ_i}, the scalar counterpart of matrix $e^{\theta \mathbf{E}_2}$, as a linear combination of the 0th and the 1st powers of λ_i:

$$e^{j\theta} = c_0 + c_1 j\theta, \quad e^{-j\theta} = c_0 - c_1 j\theta \tag{7}$$

which represents a system of two linear equations in the two unknown coefficients c_0, c_1, its solution being

$$c_0 = \cos\theta, \quad c_1 = \frac{\sin\theta}{\theta}. \tag{8}$$

Upon substitution of the foregoing coefficients into Eq. (5), $e^{\theta \mathbf{E}_2}$ becomes

$$e^{\theta \mathbf{E}_2} = \cos\theta 1_2 + \sin\theta \mathbf{E}_2 \equiv \mathbf{Q}_2 \tag{9}$$

thereby showing that $\mathbf{Q}_2$ is indeed the exponential of $\theta \mathbf{E}_2$.

In summary, the 2×2 rotation matrix is related to the complex number $u(\theta)$ via the eigenvalues of $\theta \mathbf{E}_2$, which are $u(\theta)$ and $u(-\theta)$. For completeness, the eigenvectors of $\mathbf{Q}_2$, identical to those of $\mathbf{E}_2$, are, correspondingly, $[-j, 1]^T$ and $[j, 1]^T$.

2.1. The rotation matrix in three dimensions

Prior to introducing the angular-velocity and angular-acceleration matrices in the plane, a digression is made into three dimensions. The reason is that these matrices are well known in three dimensions, while their two-dimensional counterparts are not. The purpose of this section is to make the paper self-contained.

In three dimensions, the rotation matrix $\mathbf{Q}_3$ can be expressed as a linear combination of $\mathbf{E}_3^0 \equiv 1_3$, $\mathbf{E}_3$ and $\mathbf{E}_3^2$, again, in light of the Cayley–Hamilton Theorem [7]:

$$\mathbf{Q}_3 = \mathbf{1}_3 + \sin\theta \mathbf{E}_3 + (1 - \cos\theta)\mathbf{E}_3^2, \quad \mathbf{E}_3 = \mathrm{CPM}(\mathbf{e}) \tag{10}$$

[1] More on this subject is discussed in Section 4.