Mechanics Research Communications 36 (2009) 796–803

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

# A note on the two temperature theory with dual-phase-lag delay: Some exact solutions

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ARTICLE INFO

*Article history:*
Received 18 March 2009
Available online 18 May 2009

*Keywords:*
Dual-phase-lag model
Two temperature theory of heat conduction
Lambert *W*-function

ABSTRACT

In this note we present exact solutions of two initial-boundary value problems (IBVP)s in the setting of a recently-introduced theory of heat conduction, wherein the two temperature theory of the late 1960s is merged with Tzou's dual-phase-lag flux relation. First, we solve a one-dimensional problem on a finite interval for a simple, parabolic initial condition. We then describe how to extend the analysis to the general three-dimensional case. In particular, it is demonstrated that the instability which generally arises in connection with the dual-phase-lag model can be avoided under this hybrid formulation.

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## 1. Introduction

As Maxwell (2001) appears to have been the first to point out, the classical linear theory of heat conduction, which is based on Fourier's law for the thermal flux, predicts that a thermal disturbance at some point in a material body will be felt instantly, but unequally, at all other points of the body, however distant. This behavior, which is often referred to as the "paradox of heat conduction" (Dreyer and Struchtrup, 1993), is physically unrealistic since it implies that thermal signals propagate with infinite speed. It is therefore not surprising, given this non-causal aspect of the classical theory, that numerous alternative theories of heat conduction have been put forth since Maxwell first made his observation in the latter half of the 19th century (see, e.g., Bargmann and Steinmann, 2008; Caviglia et al., 1992; Chandrasekharaiah, 1986; Chandrasekharaiah, 1998; Christov, 2009; Dreyer and Struchtrup, 1993; Hetnarski and Ignaczak, 2000; Ignaczak and Ostoja-Starzewski, 2009; Ostoja-Starzewski, 2009; Reverberi et al., 2008 and the references therein).

One of these modern theories, the so-called dual-phase-lag model, was proposed by Tzou (1995) (see also Tzou, 1997). Under this theory, Fourier's law is replaced with

$$\mathbf{q}(\mathbf{x}, t+\tau_1) = -K\nabla T(\mathbf{x}, t+\tau_2), \tag{1}$$

where $\mathbf{q}$ is the heat flux vector, $T$ is the absolute temperature, and the constant $K(>0)$ denotes the thermal conductivity. In addition, the time delay constants $\tau_1$ and $\tau_2$, where in the present note $\tau_1 > \tau_2 \geqslant 0$ will be assumed, are associated with the microstructure of the material under consideration. The dual-phase-lag model, which reduces to Fourier's law in the limit $\tau_1 - \tau_2 \to 0$, describes a process in which a temperature gradient that is established across a material volume at time $t+\tau_2$ will not give rise to a thermal flux at a point $\mathbf{x}$ within that volume until the later time $t+\tau_1$. For more on the dual-phase-lag model, see Antaki (2000), Chandrasekharaiah (1998), Horgan and Quintanilla (2005), Jou and Criado-Sancho

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0093-6413/$ - see front matter Published by Elsevier Ltd.
doi:10.1016/j.mechrescom.2009.05.002

(1998), Quintanilla (2002), Quintanilla (2003), Quintanilla and Racke (2006a), Quintanilla and Racke (2007), Shen and Zhang (2008), and the references therein.

Unfortunately, when combined with the conservation of energy law from classical heat conduction theory, namely,

$$\rho\dot{\varepsilon}(\mathbf{x},t)+\nabla\cdot\mathbf{q}(\mathbf{x},t)=0, \tag{2}$$

assuming a rigid thermal conductor and that no heat sources/sinks are present, Eq. (1) gives rise to a heat transport equation that, in general, exhibits instability with respect to the initial data. (In Eq. (2), $\dot{\varepsilon}=c_p\dot{T}$, where $\varepsilon$ denotes the specific internal energy and the specific heat at constant pressure $c_p(>0)$ is assumed constant; $\rho$ is the (constant) mass density; and a superposed dot denotes $\partial/\partial t$.) To be more specific, it can be proved that there exists a sequence of eigenvalues such that their real parts tend to infinity, meaning that Tzou's model is highly explosive Jordan et al., 2008. Thus, simply substituting the dual-phase-lag model in place of Fourier's law results in a theory that is arguably *less* realistic than the classical (i.e., Fourier-based) version!

In the late 1960s, Chen and Gurtin (1968) and Chen et al. (1968, 1969) formulated what soon became known as the two temperature theory (2TT). This theory proposes that heat conduction in a material body depends upon two distinct temperatures, the conductive temperature $\Phi$ and the thermodynamic temperature $\Theta$ (Warren and Chen, 1973). While, under certain conditions, these two temperature can be equal, in time-dependent problems, however, in particular those involving wave propagation, $\Phi$ and $\Theta$ are generally different (Warren and Chen, 1973). The key element that sets the 2TT apart from the classical theory is the material parameter $a(>0)$. Specifically, in the limit as $a\to 0$, $\Phi\to\Theta$ and the classical theory is recovered.

Although interest in the 2TT has waned since the 1970s, the recent contributions of Quintanilla (2004a,b) and Puri and Jordan (2006) has signaled something of a reversal in this trend. In particular, Quintanilla (2008) has proposed a modification of the 2TT that is based on replacing

$$\mathbf{q}(\mathbf{x},t)=-K\nabla\Phi(\mathbf{x},t). \tag{3}$$

which is the constitutive equation for the heat flux vector under the 2TT, with

$$\mathbf{q}(\mathbf{x},t+\tau_1)=-K\nabla\Phi(\mathbf{x},t+\tau_2), \tag{4}$$

which is just Eq. (1) with the conductive temperature $\Phi$ taking the place of $T$. The significance of Quintanilla's modification of the 2TT is that it yields a theory of heat conduction that includes time delay, but that does not suffer from the ill-posedness problem described by Jordan et al. (2008).

We believe that, whenever possible, one should study mathematical models in their exact form. We also believe that mathematical analysis plays a crucial role in ascertaining the validity of theoretical models, such as the one proposed by Quintanilla (2008), by uncovering bounds on the material parameters, and possibly even experimentally testable predictions. Thus, our investigation has been based on the exact, one-dimensional (1D) version of Eq. (1) – *not*, as is the case with almost all of the earlier works on this topic, its Taylor series approximation. What's more, the major findings presented here were obtained using only the methods of classical analysis; numerically-generated graphs have been included primarily to clarify the former.

Now, the present communication is organized as follows. In Section 2, we state the basic equations of the new theory. In Section 3, we solve a particular, well known, 1D IBVP. In Section 4, we sketch-out how to extend the approach of Section 3 to a general IBVP in three-dimensions. And finally, in Section 5, a summary of our findings is presented, which is followed by a short appendix on the Lambert $W$-function.

## 2. Basic equations

Along with Eq. (4), we will need the following two constitutive relations from the 2TT:

$$\Theta=\Phi-a\triangle\Phi \quad\text{and}\quad \varepsilon=\varepsilon_r+c_p(\Theta-\Phi_r), \tag{5}$$

where the constants $\varepsilon_r$ and $\Phi_r$ corresponds to the reference state. On eliminating $\Theta$ between these two equations and then applying $\partial/\partial t$ to the result, we obtain

$$\dot{\varepsilon}=c_p(\dot{\Phi}-a\triangle\dot{\Phi}). \tag{6}$$

Next, we recast Eq. (4) as

$$\mathbf{q}(\mathbf{x},t+\lambda_0)=-K\nabla\Phi(\mathbf{x},t), \tag{7}$$

where we observe that $\lambda_0=\tau_1-\tau_2$ is positive. On substituting Eqs. (6) and (7) into Eq. (2), we find that under the modified 2TT theory, the conductive temperature satisfies the (delayed) field equation

$$\dot{\Phi}(\mathbf{x},t+\lambda_0)-a\triangle\dot{\Phi}(\mathbf{x},t+\lambda_0)=\kappa\triangle\Phi(\mathbf{x},t), \tag{8}$$

where $\kappa=K(\rho c_p)^{-1}$ denotes the thermal diffusivity.

To fully define the physical problems that are to be solved, boundary and initial conditions must be imposed. In this note we restrict our attention to a bounded domain, $B$, whose boundary is smooth enough to guarantee that the conditions of the