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Response of an infinite beam resting on a tensionless elastic foundation subjected to arbitrarily complex transverse loads

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A B S T R A C T

This paper presents an investigation into the static response of an infinite beam supported on a unilateral (tensionless) elastic foundation and subjected to arbitrary complex loading, including self-weight. A new numerical method is developed to determine the initially unknown lengths that remain in contact. Based on the continuity conditions at the junctions of contact and non-contact segments, the response of the whole beam may be expressed through the displacement constants of the initial segment, reducing the contact problem to two nonlinear algebraic equations with two unknowns. The technique has been named the transfer displacement function method (TDFM). Comparison with the exact results of a particular limiting case shows the expected complete agreement. Finally, an example of a beam with several contact segments is presented and verified by the application of equilibrium conditions.

1. Introduction

The mechanical behaviour of a beam in contact with tensionless elastic foundations has attracted much attention over many years. Early studies (Weitsman, 1972) analysed the problem of an infinite beam subjected to a concentrated load, where the contact length turned out to be independent of the magnitude of the load. The problem was then extended to a dynamic model of a finite or infinite beam subjected to moving loads (Choros and Adams, 1979; Lin and Adams, 1987; Maheshwari et al., 2004) or oscillating forces (Celep et al., 1989; Coskun, 2000, 2003). To limit the number of unknowns, the systems were assumed to be symmetric and only one or two concentrated forces were considered. In 2004, an analytical model was developed by Zhang and Murphy (2004) to simulate the response of a finite beam subjected to a non-symmetric concentrated load. Kaschiev and Mikhajlov (1995) proposed a general procedure in which an iteration method combined with a finite element analysis (FEA) model was employed to analyse finite beams under arbitrary loads.

Although there is much literature relating to this class of beam contact problem, it remains a difficult task to analyse an infinite beam making unilateral contact with an elastic foundation and carrying arbitrarily complex loads. General methods combined with an FEA model might not be applicable as several contact zones may exist resulting in direct analytical methods encountering mathematical difficulties when trying to solve the corresponding large dimensional nonlinear system. In this paper, a new method named the transfer displacement function method (TDFM) is employed to determine contact and non-contact zones. Based on the continuity condition at the points separating these zones, the response of the $(i+1)$th beam segment may be expressed in terms of the displacements of the ith beam segment. As a result the displacement of the whole beam may be expressed by the response of the initial segment, reducing the contact problem to a nonlinear system with just two unknowns. In essence, the TDFM is a process to convert a problem with unknown boundary conditions into an initial

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value problem. As a technique to solve initial value problems, TDFM is similar to the transfer matrix method used for wave propagation problems in layered media.

2. Modelling of unilaterally supported infinite beams with arbitrary loading

Consider an infinite beam under complex loads including self-weight (Fig. 1), where x is the position coordinate; L_f is a locally loaded zone; q_0 is the self-weight; $v(x)$ is the deflection of the beam; k is the Winkler foundation stiffness factor; $P(x)$, $q(x)$ and $M(x)$ are the concentrated loads, distributed loads and concentrated moments acting on the beam. The infinite beam may be separated into n segments, where the separating points (shown as black dots along the *x-axis)* define locations at which contact is lost or re-established or at which there are loading or other discontinuities. It should be noted that the first and last segments are semi-infinite.

The governing equation for the ith beam segment is

$$\mathrm{EI}v_i'''' + \bar{k}v_i = q_i + q_0, \tag{1}$$

$$\text{where} \quad \bar{k} = \begin{cases} 0 & \text{non-contact,} \\ k & \text{contact,} \end{cases} \tag{2}$$

EI is the flexural stiffness of the beam, and primes denote differentiation with respect to x. The displacement satisfying (1) may be expressed as

$$v_i(x) = \bar{v}_i(x) + \tilde{v}_i(x), \tag{3}$$

where $\bar{v}_i(x)$ and $\tilde{v}_i(x)$ are the homogeneous solution and the particular solution to Eq. (1).

To avoid numerical instability for large x in later matrix operations, the homogeneous solution $\bar{v}_i$ may be expressed in terms of $(x - x_{i-1})$ for $i > 1$ (or $(x - x_1)$ for $i = 1$),

$$\bar{v}_i(x) = \begin{cases} \mathrm{e}^{\beta(x-x_1)}(a_{i,1}\cos\beta x + a_{i,2}\sin\beta x) + \mathrm{e}^{-\beta(x-x_1)}(a_{i,3}\cos\beta x + a_{i,4}\sin\beta x) & i = 1, \\ \mathrm{e}^{\beta(x-x_{i-1})}(a_{i,1}\cos\beta x + a_{i,2}\sin\beta x) + \mathrm{e}^{-\beta(x-x_{i-1})}(a_{i,3}\cos\beta x + a_{i,4}\sin\beta x) & \text{contact } i > 1, \\ a_{i,1} + a_{i,2}(x - x_{i-1}) + a_{i,3}(x - x_{i-1})^2 + a_{i,4}(x - x_{i-1})^3 & \text{non-contact,} \end{cases} \tag{4}$$

where $\beta = \sqrt[4]{k/(4EI)}$, and $a_{i,1}$, $a_{i,2}$, $a_{i,3}$, $a_{i,4}$ are coefficients to be determined.

Regarding the particular solution, for the first and the last beam segment ($i = 1$ and $i = \mathrm{n}$) where only a constant weight q_0 is considered, $\tilde{v}_1$ and $\tilde{v}_n$ may be expressed as

$$\tilde{v}_1(x) = \tilde{v}_n(x) = \frac{q_0}{4\mathrm{EI}\beta^4}. \tag{5}$$

For other beam segments ($1 < i < \mathrm{n}$), the particular solution $\tilde{v}_i$ depends on the functions defining the external distributed loads. We consider a polynomial load function

$$q_i(x) + q_0 = b_{i,0} + \sum_{k=1}^{N_{qi}} b_{i,k}(x - x_{i-1})^k, \tag{6}$$

where $b_{i,k}$ is the load coefficient and N_{qi} is the highest order of the polynomial function. Thus $\tilde{v}_i$ may be expressed as

$$\tilde{v}_i(x) = \begin{cases} \sum\limits_{k=0}^{N_{qi}} \frac{b_{i,k}(x-x_{i-1})^{k+4}}{(k+1)(k+2)(k+3)(k+4)} & \text{non-contact,} \\ \frac{1}{4\mathrm{EI}\beta^4}\left[c_{i,0} + \sum\limits_{k=1}^{N_{qi}} c_{i,k}(x - x_{i-1})^k\right] & \text{contact,} \end{cases} \tag{7}$$

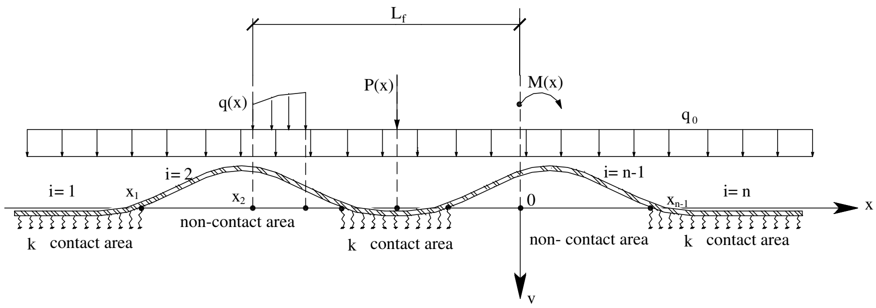

Fig. 1. An infinite beam resting on a tensionless foundation.