Available online at www.sciencedirect.com

ScienceDirect

Mechanics Research Communications 35 (2008) 276–281

MECHANICS RESEARCH COMMUNICATIONS

www.elsevier.com/locate/mechrescom

# Separation of variables in one problem of motion of the generalized Kowalevski top

M.P. Kharlamov *

*Chair of Mathematical Simulation, Volgograd Academy of Public Administration, Gagarin Street 8, Volgograd 400131, Russia*

Received 14 February 2007; received in revised form 31 January 2008
Available online 12 February 2008

**Abstract**

In the problem of motion of the Kowalevski top in a double force field the new case of reduction to a Hamiltonian system with two degrees of freedom was pointed out by Kharlamov [Kharlamov, M.P., 2004. Mekh. Tverd. Tela 34, 47–58]. We show that the equations of motion in this case can be separated by the appropriate change of variables, the new variables $U, V$ being hyperelliptic functions of time. The natural phase variables (components of the angular velocity and the direction vectors of the forces with respect to the movable basis) are expressed via $U, V$ explicitly in elementary algebraic functions.


*Keywords:* Kowalevski top; Double force field; Appelrot classes; Separation of variables

## 1. Introduction

The problem of motion of the Kowalevski top in a double force field is described by the equations

$$\begin{aligned}
&\dot\alpha_1 = \alpha_2\omega_3 - \alpha_3\omega_2, \quad \dot\beta_1 = \beta_2\omega_3 - \beta_3\omega_2, \quad \dot\omega_1 = 2^{-1}(\omega_2\omega_3 + \beta_3),\\
&\dot\alpha_2 = \alpha_3\omega_1 - \alpha_1\omega_3, \quad \dot\beta_2 = \beta_3\omega_1 - \beta_1\omega_3, \quad \dot\omega_2 = -2^{-1}(\omega_1\omega_3 + \alpha_3),\\
&\dot\alpha_3 = \alpha_1\omega_2 - \alpha_2\omega_1, \quad \dot\beta_3 = \beta_1\omega_2 - \beta_2\omega_1, \quad \dot\omega_3 = \alpha_2 - \beta_1,
\end{aligned} \tag{1}$$

and has been proved to be a completely integrable Hamiltonian system with three degrees of freedom in Bogoyavlensky (1984), Reyman and Semenov-Tian-Shanky (1989). As it is shown in Kharlamov (2005), without loss of generality this system can be considered on the common level $P^6 \subset \mathbf{R}^9(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\omega})$ of the geometrical integrals

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = a^2, \quad \beta_1^2 + \beta_2^2 + \beta_3^2 = b^2, \quad \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0. \tag{2}$$

* Tel.: +7 844 247 56 72.
*E-mail address:* mharlamov@vags.ru

0093-6413/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mechrescom.2008.02.001

The first integrals in involution are

$$\begin{aligned}
&H = \omega_1^2 + \omega_2^2 + \frac{1}{2}\omega_3^2 - (\alpha_1 + \beta_2),\\
&K = (\omega_1^2 - \omega_2^2 + \alpha_1 - \beta_2)^2 + (2\omega_1\omega_2 + \alpha_2 + \beta_1)^2,\\
&G = \frac{1}{4}(\omega_\alpha^2 + \omega_\beta^2) + \frac{1}{2}\omega_3\omega_\gamma - b^2\alpha_1 - a^2\beta_2,
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
&\omega_\alpha = 2\omega_1\alpha_1 + 2\omega_2\alpha_2 + \omega_3\alpha_3,\\
&\omega_\beta = 2\omega_1\beta_1 + 2\omega_2\beta_2 + \omega_3\beta_3,\\
&\omega_\gamma = 2\omega_1(\alpha_2\beta_3 - \alpha_3\beta_2) + 2\omega_2(\alpha_3\beta_1 - \alpha_1\beta_3) + \omega_3(\alpha_1\beta_2 - \alpha_2\beta_1).
\end{aligned}$$

In the general case

$$a > b > 0, \tag{4}$$

the system (1)–(3) has not been integrated yet. It is natural to study the invariant submanifolds in $P^6$ such that the induced system has only two degrees of freedom. It is proved in Kharlamov (2005) that there exist only three submanifolds $\mathfrak{M}, \mathfrak{N}, \mathfrak{O}$ of this type. The union $\mathfrak{M} \cup \mathfrak{N} \cup \mathfrak{O}$ coincides with the set of critical points of the integral map

$$H \times K \times G : P^6 \to \mathbf{R}^3. \tag{5}$$

In the classical Kowalevski case (Kowalevski, 1889) we must put $b = 0$. Then the critical set of (5) consists of the motions that belong to the so-called four Appelrot classes (Appelrot, 1940). The set $\mathfrak{M}$, first found in Bogoyavlensky (1984) as the zero level of the integral $K$, generalizes the 1st Appelrot class. The phase topology of the system induced on $\mathfrak{M}$ was studied in Zotev (2000). The dynamical system on $\mathfrak{N}$ generalizes the 2nd and 3rd Appelrot classes. This system was explicitly integrated in Kharlamov and Savushkin (2005).

The present work considers the restriction of the system (1) to the invariant subset $\mathfrak{O}$ defined in $P^6$ by the pair of invariant relations (Kharlamov, 2005)

$$R_1 = 0, \quad R_2 = 0, \tag{6}$$

where

$$\begin{aligned}
&R_1 = \omega_2\omega_\alpha - \omega_1\omega_\beta,\\
&R_2 = (\omega_1\omega_3 + \alpha_3)\omega_\alpha + (\omega_2\omega_3 + \beta_3)\omega_\beta + \omega_\gamma + (\alpha_2\beta_1 - \alpha_1\beta_2)\omega_3.
\end{aligned}$$

We point out the separation of variables leading to hyperelliptic quadratures and show the region of the integral constants corresponding to the solutions of (1), (2), (6).

## 2. Partial integrals

Note that the functions $R_1, R_2$ in Eq. (6) are not independent at the points

$$\omega_1 = \omega_2 = 0, \quad \alpha_3 = \beta_3 = 0. \tag{7}$$

Denote by $\Omega$ the set of points defined by (2) and (7). It is easily shown that $\Omega \subset \mathfrak{O}$ and $\Omega$ is the two-dimensional invariant submanifold of the flow (1) consisting of the pendulum motions found in Kharlamov (2005)

$$\begin{aligned}
&\boldsymbol{\alpha} = a(e_1\cos\theta - e_2\sin\theta), \quad \boldsymbol{\beta} = \pm b(e_1\sin\theta + e_2\cos\theta),\\
&\boldsymbol{\alpha} \times \boldsymbol{\beta} \equiv \pm ab e_3, \quad \boldsymbol{\omega} = \frac{\mathrm{d}\theta}{\mathrm{d}t}e_3, \quad \frac{d^2\theta}{\mathrm{d}t^2} = -(a \pm b)\sin\theta.
\end{aligned} \tag{8}$$

On such trajectories the constants of the integrals (3) satisfy one of the conditions

$$g = \mp abh, \quad k = (a \pm b)^2. \tag{9}$$