Probabilistic Engineering Mechanics 40 (2015) 1–11

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Linear backward stochastic differential systems of descriptor type with structure and applications to engineering

Bujar Gashi [a], Athanasios A. Pantelous [a,b,*]

[a] Department of Mathematical Sciences, University of Liverpool, Peach Street, L69 7ZL, UK
[b] Institute for Risk and Uncertainty, University of Liverpool, UK

ARTICLE INFO

Article history:
Received 6 May 2014
Received in revised form
3 February 2015
Accepted 11 February 2015
Available online 24 February 2015

Keywords:
Singular mass matrices
Constraint systems
Solvability
Linear stochastic descriptor equations
Exact controllability

ABSTRACT

In this paper, a class of linear stochastic differential systems of descriptor type with symmetric and skew-symmetric coefficients is considered. These kinds of systems have numerous applications in several areas of engineering, systems and control theory with potential applications to multibody systems (constrained mechanical systems with singular mass matrices), power systems, robotics and elsewhere. Thus, in our approach, using the Thompson canonical form for regular pencils, necessary and sufficient conditions for the solvability of a general class of such systems are obtained. In addition, as interesting theoretical applications, the solvability for any terminal condition and the problem of exact controllability are completely settled. An application to controlled mechanical translation systems illustrates the main findings of the paper.

1. Introduction

The dynamical behavior of many physical and engineering processes is usually modeled via different types of ordinary or partially differential equations. Particularly, if the states of these processes are in some ways constrained, like for example by *conservation laws* (such as Kirchoff's laws in electrical circuits) or by *position constraints* (such as the movement of mass points on a surface), then the appropriate mathematical model that should be used is the descriptor systems (they are known also in the literature as differential-algebraic or algebro-differential or implicit differential systems). In theory, these equations are considered to be the most general and explicit solutions that describe, for instance in mechanical engineering, the motion of constrained mechanical systems is limited by the fact that they cannot deal with singular mass matrices, see [44]. However, also in [44], examples are provided to demonstrate how systems with singular mass matrices can arise in the modeling of mechanical systems in classical mechanics, making the theory of descriptor systems of significant importance for us. Moreover, it should be pointed here that systems with singularities are not very common in classical dynamics when dealing with unconstrained motion.

Now, regarding general applications, descriptor differential systems can appear in a variety of areas of interest, such as electrical networks [14], multibody systems [13,38,39,42–44], chemical engineering [36], semidiscretized Stokes equations [3–5], multi-input multi-output economical models [11,33], etc. Moreover, some interesting examples of 2×2- descriptor systems with numerical applications have been presented in [17]. Before we proceed further, let us consider the following characteristic examples; see [25].

Example 1. A physical pendulum is modeled by the movement of a mass point with mass m in Cartesian coordinates (x, y) under the influence of gravity in a distance l around the origin. With the kinetic energy $T = m/2(\dot{x}^2 + \dot{y}^2)$ and the potential energy $U = mgy$, where g is the gravity constant, using the constraint equation $x^2 + y^2 - l^2 = 0$, we obtain the Lagrange function

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy - \lambda(x^2 + y^2 - l^2)$$

with Lagrange parameter λ. The equation of motion then have the following form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

for the variables $q = x, y, \lambda$, i.e.,

$$m\ddot{x} + 2x\lambda = 0$$
$$m\ddot{y} + 2y\lambda + mg = 0$$
$$x^2 + y^2 - l^2 = 0$$

It is clear that the system cannot have differentiation index one, it

* Corresponding author at: Department of Mathematical Sciences, University of Liverpool, Peach Street, L69 7ZL, UK.
E-mail address: a.pantelous@liverpool.ac.uk (A.A. Pantelous).

http://dx.doi.org/10.1016/j.probengmech.2015.02.003
0266-8920/

actually has differentiation index three.

Example 2. The non-stationary Stokes equation is a classical linear model for the laminar flow of a Newtonian fluid. It is described by the partial differential equation (PDE)

$$u_t = \Delta u + \nabla p, \quad \nabla \cdot u = 0,$$

together with the initial and boundary conditions. Here u describes the velocity and p the pressure of the fluid. Using the method of lines and discretizing first the space variables with finite element or difference methods typically leads to a linear descriptor system of the following form:

$$\dot{u}_h = Au_h + Bp_h, \quad B^T u_h = 0,$$

where u_h and p_h are semi-discrete approximations for u and p, respectively. If the non-uniqueness of a free constant in the pressure is fixed by the discretization method, then differentiation index is well defined for this system.

Therefore, in the literature of systems science, such types of equations have been studied intensively, theoretically as well as numerically, in recent decades. For a systematic and comprehensive exposition of the most important aspects regarding the theory and the numerical treatment of the first-order descriptor systems, the interested reader can consult [8,9,11,19,25,26,34] and the references therein. Although recently in the existing literature of descriptor systems, the generalization of linear higher-order matrix differential equations of Apostol–Kolodner type has been studied by Kalogeropoulos et al. [23] in the case of descriptor systems

$$X^{(r)}(t) = AX(t),$$

where $X(t)$ is a matrix function; see [2,6] and references therein. Furthermore, it should be mentioned here that Section 5 [45] describes a method for solving higher order equations of the following form:

$$q(D)X(t) = AX(t),$$

where q is a scalar polynomial, D is the differentiation with respect to t and A is a square matrix. Moreover, as it is known, higher order nonlinear systems, i.e.

$$F(x, \dot{x}, \ldots, x^{(n)}) = 0,$$

appear in mechanical engineering, and consequently linear higher order descriptor systems can arise naturally from the linearization process of these systems, see for the case of second-order nonlinear systems [40]. Furthermore, in some of those applications, it is consistent to design the coefficient matrices of the model by considering a special structure (i.e. with symmetric or skew-symmetric properties, etc.), see [43]. Particularly, by following this direction, in [24,34], the solution for higher order linear descriptor (matrix) differential systems of Apostol–Kolodner type is investigated by considering pairs of complex matrices with symmetric and skew symmetric structural properties.

Theory for linear descriptor systems in a stochastic framework has been developed only lately in [1,21], where a special class of linear stochastic descriptor systems with delays, constant coefficients, external differentiable and non-differentiable perturbations is considered. Using the regular Matrix Pencil theory, see [19], the form of the initial function is given, so the corresponding initial value problem is uniquely solvable using the theory of generalized stochastic processes, see also [1]. Finally, it should be mentioned that two illustrative applications are presented in [21] using white noise and fractional white noise. More recently, in [16], the basic question of solvability has formulated and considered. There, under the assumption that the pencil is regular, but without assuming any particular structure for the matrices, and using fundamental tools of matrix pencil theory, such as the Weierstrass complex canonical form, the necessary and sufficient conditions for the existence of a unique solution pair are derived. Moreover, a normalization procedure is proposed, and the problem of exact controllability for a class of linear descriptor stochastic systems is completely settled.

In the present paper, a general class of linear stochastic differential systems of descriptor type with symmetric and skew-symmetric coefficients is considered. These systems are derived from the linearization process of higher order nonlinear systems in mechanical engineering, and the singular mass matrix is constant as it has been discussed above, see also about the transformation of high order linear descriptor systems to first order [23,31,35] and references therein. Thus, the final value problem (backward case) is proposed and examined. It is proved that this is uniquely solvable and the solution is derived analytically. In our approach, using the Thompson canonical form for regular pencils, necessary and sufficient conditions for the solvability of a general class of such systems are obtained. In addition, the solvability for any terminal condition and the problem of exact controllability are completely settled.

The paper is organized as follows. In Section 2, some known but necessary preliminary results on the Thompson canonical form are presented for readers convenience. In Section 3 we consider the problem of solvability. This is done in three stages. We begin with two special equations, which are then used to deal with the general case. In Section 4, as an interesting theoretical application we address the problem of exact controllability for linear stochastic systems of descriptor type with special structure, symmetric/skew-symmetric case. Moreover, a numerical example is presented to illustrate further the main findings of the paper. Finally, Section 5 concludes the whole discussion providing new directions for research.

2. Model formulation

Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ be a given and complete filtered probability space, on which a scalar standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathcal{F}_t$ is the augmentation of $\sigma\{W(s)|0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. If $\xi: \Omega \to \mathbb{R}^n$ is an $\mathcal{F}_T$-random variable such that $\mathbb{E}[|\xi|^2] < \infty$, we write $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$. Let $f: [0, T] \times \Omega \to \mathbb{R}^n$ denote an $\{\mathcal{F}_t\}_{t\geq 0}$ adapted process; if $\mathbb{E}\int_0^T |f(t)|^2\, dt < \infty$, we write $f(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$; if $f(\cdot)$ has a.s. continuous sample paths and $\mathbb{E}\sup_{t\in[0,T]}|f(t)|^2 < \infty$, we write $f(\cdot) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}^n))$.

Linear backward stochastic differential equations (BSDEs) of *descriptor type* without structural coefficients were introduced by Gashi and Pantelous in their recent paper [16] extending further the ideas and the results presented in [1,20] which were based on generalized stochastic processes of Dirac type. In that paper, the main equation is given by

$$\begin{cases} E\, dx(t) = [Ax(t) + Bu(t) + Cz(t)]\, dt + z(t)\, dW(t), \\ x(T) = \xi, \quad a.s. \end{cases} \tag{2.1}$$

where E is a singular matrix, i.e. $detE=0$, $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$, and $u(\cdot) \in L^2_{\mathcal{F}}\left(0, T; \mathbb{R}^m\right)$.

The problem of solvability for these equations is that of existence of a solution pair $(x(t), z(t))$. In their non-descriptor form, i.e. when $E=I$, these have a long and fruitful history. They were first introduced by Bismut in [7] in the context of stochastic linear-