Probabilistic Engineering Mechanics 40 (2015) 90–96

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

New method for efficient Monte Carlo–Neumann solution of linear stochastic systems

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ARTICLE INFO

Article history:
Received 29 July 2014
Received in revised form
5 January 2015
Accepted 11 February 2015
Available online 7 April 2015

Keywords:
Stochastic systems
Neumann series
Monte Carlo simulation
Stochastic processes
Optimization

ABSTRACT

The Neumann series is a well-known technique to aid the solution of uncertainty propagation problems. However, convergence of the Neumann series can be very slow, often turning its use highly inefficient. In this article, a λ convergence parameter is introduced, which yields accurate and efficient Monte Carlo–Neumann solutions of linear stochastic systems using first order Neumann expansions. The λ convergence parameter is found as solution to a distance minimization problem, for an approximation of the inverse of the system matrix using the Neumann series. The method presented herein is called Monte Carlo–Neumann with λ convergence, or simply MC–N λ method. The accuracy and efficiency of the MC–N λ method is demonstrated in application to stochastic beam bending problems.

1. Introduction

The last few decades have witnessed tremendous developments in the modeling of mechanical and structural systems, due to advances in computational mechanics. Numerical methods such as finite elements, finite difference, boundary elements, etc., have reached broad acceptability and wide coverage of applications. New developments address the solution of complex, non-linear problems. Multi-physics analyses allow investigation of new, unforeseen interaction effects between structures, soils, fluids, thermo-dynamic and electric effects. Significant developments have also been recently achieved in modeling uncertainty propagation through mechanical and other types of systems.

The Monte Carlo simulation method remains a popular, yet computationally expensive tool for analyzing uncertainty propagation through mechanical systems. The computational cost of Monte Carlo simulation can easily become prohibitive, for highly non-linear problems and complex geometries. More efficient, intrusive methods have recently been developed, such as the stochastic finite element method [1] or stochastic Galerkin Method [2–6]. Intrusive methods have the inconveniency of requiring full re-programming of conventional finite element software. Hence, non-intrusive Monte Carlo simulation methods remain popular in the solution of stochastic mechanics problems.

In linear stochastic mechanics problems, the numerical solution of a differential equation is replaced by the solution of a linear system of algebraic equations (stiffness matrix). In this context, when Monte Carlo simulation is employed, for each system realization, the stiffness matrix needs to be evaluated and inverted. Depending on the dimensions of the linear system, and the required number of samples, this can become computationally intensive. For a linear operator in finite dimensions, $\mathcal{K}$: $\mathbb{R}^n \rightarrow \mathbb{R}^n$, the inverse can be represented by the Neumann series, composed of operators $\mathcal{P}$: $\mathbb{R}^n \rightarrow \mathbb{R}^n$related to $\mathcal{K}$. In finite dimensions, linear operators are matrices. The objective of using the Neumann series is to replace the matrix inversions by a truncated series expansion. However, depending on the number of terms in the Neumann series, the number of operations to be performed may become larger than required for the direct linear system solution. Therefore, in this paper, a λ convergence parameter is introduced, with allows accurate solutions to be computed using very low order Neumann expansions.

First use of the Neumann series to solve stochastic problems in mechanics goes back to the late eighties. Yamazaki [7] employed the Neumann series to obtain samples of the displacement response, for a plane elasticity problem with random Youngs modulus. Araújo and Awruch [8] derived the response on non-linear static and dynamic problems, with random mechanical properties. Also using the Neumann series, Chakraborty and Dey [9,10] obtained expected value and variance of displacement responses considering uncertain geometries and mechanical properties. Chakraborty and Dey [11] applied the Neumann series to solve, in

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http://dx.doi.org/10.1016/j.probengmech.2015.02.006
0266-8920/

the frequency domain, two dynamic plate bending problems involving uncertain parameters. Lei and Qiu [12] employed the Neumann series to study uncertainty propagation in structures using movement equations. Chakraborty and Sarkar [13] estimated statistical moments of the transversal displacement response of curved beams resting on Winkler foundations. Chakraborty and Bhattacharyya [14] obtained response statistics for linear tri-dimensional elasticity problems, representing elastic properties as Gaussian processes of the continuous. Li et al. [15] presented a methodology to obtain solutions of linear systems, based on a discretization of the stochastic problem. Schevenels et al. [16] proposed a methodology based on Green functions, which was compared to the Neumann series, in a wave propagation problem on random Winkler foundation. In all references above, the Neumann series was employed in a conventional fashion, as an alternative to solve the stochastic problem. No studies have been found in the literature addressing the efficiency of the Neumann expansion, or trying to improve its accuracy using low order expansions.

This article advances the state of the art by proposing a λ convergence parameter, with significantly improves accuracy of matrix inversion using low (first) order Neumann expansions. Being of low order, such solutions are also significantly more efficient than conventional Monte Carlo or conventional, higher order Neumann expansions. The accuracy and efficiency of the proposed Monte Carlo–Neumann with λ convergence, or simply MC–N λ method, are demonstrated in application to stochastic beam bending problems.

2. Mathematical formulation of the stochastic system/uncertainty propagation problem

The formulation of many problems of mechanics derives from well-known physical principles such as conservation of movement, first and second laws of thermodynamics, among other. In linear solid mechanics, equilibrium equations are often derived from the minimization of functionals. One of the best known is the Principle of Minimal Potential Energy, for which the functional minimization yields the Euler–Lagrange equations, and the essential and natural boundary conditions. In this paper, a linear elliptic boundary value problem is addressed, whose operator appears in beam and plate bending problems, or in stationary head conduction problems. The stochastic uncertainty propagation problem is defined in a $(\Omega, \mathcal{F}, P)$ probability space, where Ω is the sample space, $\mathcal{F}$ is a σ-algebra of events, and P is a probability measure.

The linear stochastic uncertainty propagation problem consists in finding the response process for the following elliptic problem, with coefficients given by stochastic processes

$$\begin{cases} \text{Find} u \in L^2\Big(\Omega, \mathcal{F}, P; \big(H^{2m}(D) \cap H_0^m(D)\big)\Big), \text{ such that} \\ \sum\limits_{|\alpha|,|\beta| \le m} \partial_\alpha(\kappa_{\alpha\beta}\partial_\beta)(x, \omega) \\ \quad = f(x, \omega), \forall\, (x, \omega) \in D \times (\Omega, \mathcal{F}, P),\ a.\ e.\ ; \\ \text{subject to boundary conditions.} \end{cases} \tag{1}$$

where $f(\cdot, \cdot)$ is a source term. The random character of the solution is given by the set of coefficients $\{\kappa_{\alpha\beta}\}_{|\alpha|,|\beta|<m}$. In solid mechanics problems, these coefficients can be associated to stiffness or thermal conductivity. In order to establish the conditions for existence and uniqueness of the solution to the problem stated in Eq. (1), the following hypothesis are required:

$$\begin{aligned} &H1\colon \kappa_{\alpha\beta} \in L^\infty\big(\Omega, \mathcal{F}, P; H^m(D)\big), \\ &\kappa_{\alpha\beta} \ge 0, \quad \text{a.e.em } D, \forall\, \alpha, \beta \in \mathbb{N}^m; \\ &H2\colon f \in L^2\Big(\Omega, \mathcal{F}, P; L^2(D)\Big). \end{aligned} \tag{2}$$

Hypothesis H1 ensures that coefficients $\{\kappa_{\alpha\beta}\}_{|\alpha|,|\beta|<m}$ be differentiable, positive and uniformly limited in probability [17]. Hypothesis H2 ensures that the source term has finite variance. From Eq. (2), the Lax-Milgram lemma can be called upon, in order to guarantee existence and uniqueness of the solutions or system response realizations, for samples of random coefficients $\{\kappa_{\alpha\beta}\}_{|\alpha|,|\beta|<m}$ and of the source term. A formal study of existence and uniqueness of the solutions is out of scope for this paper, but can be found in Refs. [2–6,18].

The great benefit of employing numerical methods in the solution of problems like Eq. (1) is reducing problem complexity. In general, solution of a stochastic differential equation is replaced by the solution of a system of algebraic equations. The Galerkin method is one of the most popular. Numerical solutions are derived from the Abstract Variational Problem (AVP) derived from Eq. (1). For the kth sample $\Big\{\kappa_{\alpha\beta}\Big(x, \xi_{\alpha\beta}(\omega_k)\Big)\Big\}$, the AVP is defined in $V = H_0^m(D)$ and is given by

$$\begin{cases} \textit{For } \big\{\xi_{\alpha\beta}(\omega_k)\big\} \textit{ fixed, find } u \in V \textit{ such that}: \\ a(u, v) = l(v), \forall\, v \in V; \end{cases} \tag{3}$$

where $a(\cdot, \cdot)$ is a bilinear form and $l(\cdot)$ is a linear functional, defined as

$$\begin{cases} a(u, v) = \sum\limits_{|\alpha|,|\beta| \le m} \int_D \kappa_{\alpha\beta}\Big(x, \xi_{\alpha\beta}(\omega_k)\Big) \cdot \partial_\alpha u(x, \omega_k) \partial_\beta v(x) dx, \\ l(v) = \int_D (f \cdot v)(x, \omega) dx + \text{countour integrals}. \end{cases} \tag{4}$$

Hence, for a fixed sample of coefficients $\Big\{\kappa_{\alpha\beta}\Big(x, \xi_{\alpha\beta}(\omega_k)\Big)\Big\}$, solution of the AVP corresponds to one realization of the operator defined in Eq. (1). From the AVP, the Galerkin solution is constructed. The approximation space $V_m = span\{\varphi_1, \ldots, \varphi_m\}$ is obtained from a subset of V, $\Big(V = \overline{span\{\varphi_i\}_{i \in \mathbb{N}}}^V\Big)$. Without loss of generality, the source term is considered deterministic in this paper. For each sample of coefficients, solutions to the AVP are measurable; and from the Doob–Dynkin lemma, the response process depends on the random variables which describe uncertainty in the coefficients; hence $u = u\Big(x, \xi_{\alpha\beta}(\omega_k)\Big)$. For the kth realization of system coefficients, the Galerkin method yields approximated numerical solutions (u_m), for the AVP in Eq. (3)

$$u_m\Big(x, \xi_{\alpha\beta}(\omega_k)\Big) = \sum_{i=1}^{m} u_i\Big(\xi_{\alpha\beta}(\omega_k)\Big)\varphi_i(x), \tag{5}$$

where $\Big\{u_i\Big(\xi_{\alpha\beta}(\omega_k)\Big)\Big\}_{i=1}^m$ are coordinate functions.

For the kth sample, the approximate variational problem is defined as

$$\begin{cases} \textit{For fixed } \big\{\xi_{\alpha\beta}(\omega_k)\big\}, \textit{ find } u_m \in V_m \textit{ such that}: \\ a(u_m, v) = l(v), \quad \forall\, v \in V_m. \end{cases} \tag{6}$$

By replacing Eq. (6) in Eq. (3), a linear system of algebraic equations is obtained

$$\begin{cases} \textit{For fixed } \big\{\xi_{\alpha\beta}(\omega_k)\big\} \textit{ find } \mathcal{U}\Big(\xi_{\alpha\beta}(\omega_k)\Big) \in \mathbb{R}^m \textit{ such that}: \\ \mathcal{K}\Big(\xi_{\alpha\beta}(\omega_k)\Big)\mathcal{U}\Big(\xi_{\alpha\beta}(\omega_k)\Big) = \mathcal{F}; \end{cases} \tag{7}$$