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Evaluating correlation coefficient for Nataf transformation

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ABSTRACT

In this paper, a novel approach is proposed to calculate the equivalent correlation coefficient ρ_z in the standard normal space for two correlated random variables with desired correlation coefficient ρ_x. According to Weierstrass approximation theorem, ρ_x is expressed as a polynomial function of ρ_z. For a given ρ_x, the associated ρ_z is evaluated by solving the polynomial equation. Especially, when one random variable is normal, ρ_x is proved to be a linear function of ρ_z. In order to check the proposed method, a Monte Carlo simulation method is put forward. Finally, three numerical examples are worked to demonstrate the proposed method.

1. Introduction

For analysis of engineering systems under uncertainty, the input variables are preferable to be modeled as a random vector with each entry following respective probability distribution, that is, $\boldsymbol{X} = (x_1, \ldots, x_i, \ldots, x_m)^T$. To make the problem more tractable, a transformation of $\boldsymbol{X}$ into independent standard normal space is often invoked. If the joint probability density function (PDF) of $\boldsymbol{X}$ is known, Rosenblatt transformation [1] is viable. While, this method leads to $m!$ different transformations according to the ordering of the inputs, and not all of them give rise to the same favorable numerical properties. Moreover, the joint PDF is seldom available in many practical applications. Thus, the Nataf transformation [2] makes a worthwhile alternative to normalize the inputs, which requires the marginal PDFs and the correlation matrix of the input random variables.

The major obstacle for Nataf transformation is to evaluate the equivalent correlation matrix in the standard normal space. More specifically, it requires to evaluate the correlation coefficient ρ_z in the standard normal space for the correlation coefficient ρ_x of two correlated random variables. Heretofore, much research has been done to solve this problem, such as the root finding method [3,4], the empirical formulae given for commonly used distributions [2,5], the artificial neural method [6], the linear search method [7] and the polynomial normal transformation technique [8,9]. Among these approaches, the empirical formulae and the linear search method stand out for the generality and efficiency. Through numerical experiment, 49 empirical formulae for 10 kinds of distributions have been given. Although accurate results are obtained by these formulae, not all probability distributions are covered. Li et al. propose a linear search method to find the value of ρ_z, of which the efficiency is slightly lower than the empirical formula [7].

This paper is devoted to the evaluation of ρ_z for random variables with arbitrary marginal distributions. According to Weierstrass approximation theorem, the original correlation coefficient ρ_x is expressed as a polynomial function of ρ_z. For two arbitrary probability distributions, the function relationship between ρ_x and ρ_z can be establish by performing the interpolation only once. The value of ρ_z is easily obtained by solving the polynomial equation. To verify the proposed method, a Monte Carlo simulation (MCS) for calculating ρ_z is also presented.

2. Nataf transformation

The basic idea of Nataf transformation is to generate correlated random vector with a specified correlation matrix from independent standard normal deviates. This method is also known as the NORTA (NORmal To Anything) algorithm [10]. The transformation from standard normal deviates to random variables x with a specified cumulative distribution function (CDF) $F(x)$ is as follows:

$$F(x) = \Phi(z) \tag{1}$$

$$x = F^{-1}[\Phi(z)] \tag{2}$$

where $\Phi(\cdot)$ is the CDF of the standard normal variable z, $F^{-1}(\cdot)$ is the inverse CDF of x.

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Suppose $\boldsymbol{X}=(x_1,\ldots,x_i,\ldots,x_m)^T$ is a random vector with correlation matrix

$$\boldsymbol{R_X}=\begin{pmatrix}\rho^x_{1,1} & \cdots & \rho^x_{1,j} & \cdots & \rho^x_{1,m}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \rho^x_{i,1} & \cdots & \rho^x_{i,j} & \cdots & \rho^x_{i,m}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \rho^x_{m,1} & \cdots & \rho^x_{m,j} & \cdots & \rho^x_{m,m}\end{pmatrix}$$

Based on the transformation in Eq. (2), $\boldsymbol{X}$ can be generated by a correlated standard normal vector $\boldsymbol{Z}=(z_1,\ldots z_i,\ldots z_m)^T$ with correlation matrix

$$\boldsymbol{R_Z}=\begin{pmatrix}\rho^z_{1,1} & \cdots & \rho^z_{1,j} & \cdots & \rho^z_{1,m}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \rho^z_{i,1} & \cdots & \rho^z_{i,j} & \cdots & \rho^z_{i,m}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \rho^z_{m,1} & \cdots & \rho^z_{m,j} & \cdots & \rho^z_{m,m}\end{pmatrix}$$

As long as the correlation matrix $\boldsymbol{R_Z}$ is known, $\boldsymbol{Z}$ can be easily generated by the following linear transformation:

$$\boldsymbol{U}=\boldsymbol{L}^{-1}\boldsymbol{Z}\leftrightarrow\boldsymbol{Z}=\boldsymbol{L}\boldsymbol{U} \tag{3}$$

where $\boldsymbol{U}=(u_1,\ldots u_i,\ldots u_m)^T$ is an independent standard normal vector, $\boldsymbol{L}$ represents the lower triangular matrix obtained from Cholesky decomposition of

$$\boldsymbol{R_Z}=\boldsymbol{L}\boldsymbol{L}^T \tag{4}$$

The procedures of generating correlated random vector from standard normal deviates can be expressed as follows:

$$\begin{pmatrix}u_1\\ \vdots\\ u_i\\ \vdots\\ u_m\end{pmatrix}\xrightarrow[z=Lu]{R_Z=LL^T}\begin{pmatrix}z_1\\ \vdots\\ z_i\\ \vdots\\ z_m\end{pmatrix}\xrightarrow{x_i=F_i^{-1}[\Phi(z_i)]}\begin{pmatrix}x_1\\ \vdots\\ x_i\\ \vdots\\ x_m\end{pmatrix} \tag{5}$$

The principal problem for Nataf transformation is to determine a suitable correlation matrix $\boldsymbol{R_Z}$ in the normal space, such that a desired correlation matrix $\boldsymbol{R_X}$ of $\boldsymbol{X}$ is guaranteed. More specifically, it requires to calculate $\rho_z(i,j)$ $(i\neq j)$ of $\boldsymbol{R_Z}$ for each entry $\rho_x(i,j)$ of $\boldsymbol{R_X}$.

3. The function relationship between ρ_x and ρ_z

Suppose x_i, x_j are two correlated random variables with correlation coefficient ρ_x, which are generated by two correlated standard normal random variables z_i and z_j respectively. Let ρ_z denote the correlation coefficient between z_i and z_j. Using the first cross product moment of x_i and x_j, the functional relationship between ρ_x and ρ_z is established:

$$\begin{aligned}\rho_x\sigma_i\sigma_j+\mu_i\mu_j&=E[x_ix_j]\\ &=\iint F_i^{-1}[\Phi(z_i)]F_j^{-1}[\Phi(z_j)]\phi(z_i,z_j,\rho_z)\,dz_i\,dz_j\end{aligned} \tag{6}$$

where $\phi(z_i,z_j,\rho_z)$ is the joint PDF of two correlated standard normal variables. μ_i, μ_j denote the means of x_i, x_j respectively, σ_i, σ_j denote the standard deviations respectively.

For the two dimensional standard normal vector $\boldsymbol{Z}=(z_i,z_j)^T$, the correlation matrix is

$$\boldsymbol{R_Z}=\begin{pmatrix}1 & \rho_z\\ \rho_z & 1\end{pmatrix}$$

The lower triangular matrix from Cholesky decomposition of $\boldsymbol{R_Z}$ is

$$\boldsymbol{L}=\begin{pmatrix}1 & 0\\ \rho_z & \sqrt{1-\rho_z^2}\end{pmatrix}$$

Via the procedures in Eq. (5), the two correlated random variable x_i and x_j are generated by

$$\begin{aligned}x_i&=F_i^{-1}[\Phi(u_i)]\\ x_j&=F_j^{-1}[\Phi(\rho_zu_i+\sqrt{1-\rho_z^2}u_j)]\end{aligned} \tag{7}$$

where u_i and u_j are mutually independent standard normal variables.

The joint PDF of u_i and u_j is

$$\phi(u_i,u_j)=\frac{1}{2\pi}e^{-(u_i^2+u_j^2)/2} \tag{8}$$

Then, $E[x_ix_j]$ is calculated as

$$E[x_ix_j]=\frac{1}{2\pi}\iint F_i^{-1}[\Phi(u_i)]\cdot F_j^{-1}[\Phi(\rho_zu_i+\sqrt{1-\rho_z^2}u_j)]e^{-(u_i^2+u_j^2)/2}\,du_i\,du_j \tag{9}$$

Eq. (6) is rewritten as

$$\rho_x=-\frac{\mu_i\mu_j}{\sigma_i\sigma_j}+\frac{1}{2\pi\sigma_i\sigma_j}\iint F_i^{-1}[\Phi(u_i)]\quad\cdot F_j^{-1}[\Phi(\rho_zu_i+\sqrt{1-\rho_z^2}u_j)]e^{-(u_i^2+u_j^2)/2}\,du_i\,du_j \tag{10}$$

The above equation defines the functional relationship between ρ_z and ρ_x:

$$\rho_x=G(\rho_z) \tag{11}$$

If the integral on the right-hand side of Eq. (10) can be carried out analytically, $\rho_x=G(\rho_z)$ can be easily obtained. However, in general, the integral is not tractable [3]. While, for three cases, the analytical expressions of $G(\cdot)$ are obtainable, which are presented in Appendix A.

However, some properties of the function $G(\cdot)$ have been proved [2], which enable us to perform an efficient numerical method to find the value of ρ_z.

Lemma 1. *ρ_x is a strictly increasing function of ρ_z.*

Lemma 2. *$\rho_z=0$ for $\rho_x=0$.*

Lemma 3. *$|\rho_x|\leq|\rho_z|$.*

4. Monte Carlo simulation method

This section presents a MCS method to evaluate ρ_z for a specified ρ_x. According to Eq. (6), ρ_x is expressed as

$$\rho_x=\frac{E[x_ix_j]-\mu_i\mu_j}{\sigma_i\sigma_j}=Corr(x_i,x_j) \tag{12}$$

The basic idea of the MCS method is straightforward. Lemma 1 indicates that $G(\cdot)$ is a strictly increasing function, thus, there is a one to one correspondence between ρ_x and ρ_z. For a given value of ρ_x between x_i and x_j, ρ_z can be assessed by the following steps:

(1) Select a set value of ρ_z^k evenly spaced across the interval $[-1,1]$ in steps of $\Delta\rho_z$.
(2) For each ρ_z^k, generate N bivariate random vectors $(x_i,x_j)^T$ by the transformation in Eq. (7), evaluate the correlation coefficient of the samples, obtaining ρ_x^k.
(3) For a prescribed ρ_x, find the interval $[\rho_x^k,\rho_x^{k+1}]$, where ρ_x is located, i.e. $\rho_x^k\leq\rho_x\leq\rho_x^{k+1}$. Then, ρ_z is contained in the interval