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# A weak energy–momentum method for stochastic instability induced by dissipation and random excitations

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ABSTRACT

This paper focuses on the problem of stochastic instability resulting from the action of dissipation and random excitations. The energy–momentum theorem is extended from deterministic Hamiltonian systems to stochastic Hamiltonian systems, and then a weak energy–momentum method is presented for stochastic instability analysis of random systems suffering destabilizing effects of dissipation and random excitations. The presented method combines the stochastic averaging procedure to formulate the equivalent systems of random systems for obtaining the stochastic instability criteria in probability, and can be applied to a class of systems including random gyroscopic systems with positive or negative definite potential energy. As an example, the stochastic instability conditions of a Lagrange top subjected to random vertical support excitations are formulated to express the stochastic instability induced by dissipation and random excitations.



## 1. Introduction

The concept of a dissipation-induced instability was derived from the classical Treatise on Natural Philosophy of Thomson and Tait [1], whose results were later proved by Chetayev [2] and advanced by Merkin [3]. In the past three decades, the dissipation-induced instability phenomena have been investigated in many literatures [4–16]. A multitude of physical examples and applications in those literatures has demonstrated that the effect of dissipation is one of the paramount ones governing instability mechanisms in nature, and indicated that the determination of the stability of relative equilibriums of dissipated Hamiltonian systems is a central and interested problem. Marsden et al. [4], Simo et al. [5] and Marsden [6] have presented a general approach to the stability analysis of relative equilibriums in Hamiltonian systems. This approach, which was referred to as an energy–momentum theorem/method, overcomes the deficiency of the energy–Casimir method [17–19]. The energy–momentum method is based on the use of the augmented Hamiltonian (Hamiltonian plus a conserved quantity). One can think of this method as a synthesis of the ideas of Arnold for the group variable, and those of Routh and Smale for the internal variables [7]. A series of papers [7–9,11–13] developing and applying the energy–momentum method have illustrated that this method has a nontrivial impact on the modern approach to mechanics including dynamical systems theory and geometry.

In all the investigations discussed above, the deterministic system was employed to analyze the dissipation-induced instabilities. The energy–momentum method is also aimed at the deterministic Hamiltonian system. However, random excitations are existing and inevitable in many engineering and physical applications. It is well known that the influence of random excitations plays an important role in system stability analysis. Thus, a study on the stochastic instability induced by both dissipation and random excitations is significant and necessary. It indicates that the energy–momentum theorem/method needs to be extended from deterministic Hamiltonian systems to stochastic Hamiltonian systems in order to study the stochastic instability of systems suffering destabilizing effects of dissipation and random excitations.

Gyroscopic systems, as main research targets in dissipation-induced instability problems, find wide usage in engineering applications. The stochastic stability of gyroscopic systems has been investigated by some researchers [20–24]. Namachchivaya [20] studied the mean square stability of a gyropendulum under random vertical support excitations with the aid of the stochastic averaging procedure. Using the procedure established by Namachchivaya et al. [25], the asymptotic expansion for the maximal Lyapunov exponent of a stochastic gyroscopic system was calculated and a rotating shaft system subjected to white noise excitations was analyzed in Ref. [21]. Zhu [26] and Zhu et al. [27]

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proposed a stochastic averaging method for quasi-integrable-Hamiltonian systems and combined it with Khasminskii's procedure to formulate the asymptotic expression of the largest Lyapunov exponent for investigating the stochastic stability of quasi-integrable-Hamiltonian systems. Then the presented method was applied to evaluate the largest Lyapunov exponent of a gyropendulum subjected to random vertical support-excitation in Ref. [22]. The moment stability and almost-sure stability of a stochastic gyroscopic system were obtained and applied to study the stochastic stability of a pipe conveying pulsating fluid by Vedula et al. [23]. Wang et al. [24] employed an asymptotic approach based on the stochastic averaging method to investigate the mean square stability of a MEMS gyroscope subjected to stochastic angular rates. However, in the above studies on stochastic gyroscopic systems, the prerequisite on stochastic stability analysis is that the system matrix representing the terms arising from the potential energy is positive definite. In other words, the existing studies aim at a stochastic gyroscopic system with positive definite potential energy as the precondition. Therefore, it is difficult to employ those mentioned methods in studying stochastic stability of a stochastic gyroscopic system with negative definite potential energy. This raises the need for the development of stochastic stability theorem and method.

In this paper, the energy–momentum theorem/method is extended to stochastic Hamiltonian systems, and then a weak energy–momentum method is presented for stochastic instability analysis. This method, which overcomes the deficiency of the existing approaches, can be used to analyze the stochastic instability in probability of random gyroscopic systems with positive or negative definite potential energy. With the aid of the presented method, a Lagrange top under the destabilizing effects of dissipation and random excitations are analyzed as an example.

## 2. The weak energy–momentum method

The energy–momentum theorem/method [4–6] is introduced to discuss the dissipation-induced instability of deterministic systems. In studying stochastic instability induced by both dissipation and random excitation, a natural step involves the extension of the energy–momentum theorem. Thus, a more general theorem originating from the energy–momentum theorem is accounted, and a new method, dubbed as the weak energy–momentum method, for stochastic instability analysis is presented in this section.

### 2.1. The extended energy–momentum theorem

To illustrate the effect of dissipation and random excitations on system stochastic instability, we study a stochastic Hamiltonian system with the aid of a synthesis of the energy–momentum theorem [4–6,12] and the theory of stochastic Hamiltonian systems [28]. Consider a stochastic Hamiltonian system on a configuration manifold $Q$ with elements denoted by $\boldsymbol{q} \in Q$, and canonical phase space $P$, which is the cotangent bundle $P = T^*Q$. The pair $(\boldsymbol{q}, \boldsymbol{p}) \in QT_q^*Q$ of the canonical cotangent coordinates is identified with $\boldsymbol{z} \in T^*Q$ where $\boldsymbol{p} \in T_q^*Q$ is the associated momentum. The tangent space $T_qQ$ and $T_q^*Q$ are in duality via a non-degenerate paring denoted by $\langle \cdot, \cdot \rangle$. Assume that the stochastic Hamiltonian system, with the Hamiltonian function denoted by $H : P \to \mathbb{R}$, possesses symmetry induced by a Lie group $G$ with a Lie algebra $\mathfrak{I}$, which acts on $P$ by canonical transformations.

It is defined that a function $\boldsymbol{\Theta}(\xi)$ is said to be a strongly (respectively, weakly) conserved quantity of a stochastic Hamiltonian system if for each continuous symmetry $\xi \in \mathfrak{I}$ we have that $\boldsymbol{\Theta}(\xi) = \boldsymbol{\Theta}(\xi_0)$ (respectively, $E[\boldsymbol{\Theta}(\xi)] = E[\boldsymbol{\Theta}(\xi_0)]$, for any stopping time) where $E[\cdot]$ is the expectation operator [28]. If the stochastic Hamiltonian system possesses a strongly (respectively, weakly) conserved quantity $\boldsymbol{\Theta}(\xi)$ with the same dimension as the group $G$ and $\mathfrak{I}^*$ defines as the dual of the Lie algebra $\mathfrak{I}$, a momentum map is defined by $\boldsymbol{\Theta} : P \to \mathfrak{I}^*$ for the action of $G$ on $P$, which reproduces as special cases the usual angular and liner momentums.

Consider a relative equilibrium $\boldsymbol{z}_e \in P$, $\mu = \boldsymbol{\Theta}(\boldsymbol{z}_e)$; thus, there is a $\xi \in \mathfrak{I}$ such that $\boldsymbol{z}_e$ is a critical point of the augmented Hamiltonian $H_\xi(\boldsymbol{z}) = H(\boldsymbol{z}) - \langle \boldsymbol{\Theta}(\boldsymbol{z}) - \mu, \xi \rangle$, i.e., $\delta H_\xi(\boldsymbol{z}_e) = 0$. This is same as $\boldsymbol{z}_e$ being a critical point of the energy–momentum map $H\boldsymbol{\Theta} : P \to \mathbb{R}\mathfrak{I}^*$. Choose a subspace $\mathbf{S} \subset \ker D\boldsymbol{\Theta}(\boldsymbol{z}_e)$ that is transverse to the $G_\mu$ orbit within $\ker D\boldsymbol{\Theta}(\boldsymbol{z}_e)$. According to the stochastic Dirichlet's criterion [28], the energy–momentum theorem [4–6] can be extended as follows: If the second variation of the augmented Hamiltonian, $\delta^2 H_\xi(\boldsymbol{z}_e)$, is definite on $\boldsymbol{S}$ and $H_\xi$ is a strongly (respectively, weakly) conserved quantity, then $\boldsymbol{z}_e$ is $G_\mu$-orbitally almost surely stable (respectively, stable in probability) in $\boldsymbol{\Theta}^{-1}(\mu)$ and $G$-orbitally almost surely stable (respectively, stable in probability) in $P$. If $\delta^2 H_\xi(\boldsymbol{z}_e)$ is indefinite, then the relative equilibrium gets destabilized after the addition of dissipation.

Notice that the definitions of strongly and weakly conserved quantities coincide for deterministic systems with the standard definition of conserved quantity. The energy–momentum theorem [4–6] for deterministic systems is consistent with the proposed theorem related to strongly conserved quantities. Therefore, the extended theorem mentioned above is a more general energy–momentum theorem.

In generally, stochastic Hamiltonian systems possess weakly conserved quantities. It is important to discuss the extended energy–momentum theorem related to weakly conserved quantities, and use it for developing an analytical method for stochastic instability analysis. Therefore, the new method, grounded on the extended energy–momentum theorem, would be dubbed as the weak energy–momentum method. Its analytic procedure for stochastic instability analysis of gyroscopic systems is illustrated in the following.

### 2.2. Stochastic instability induced by dissipation and random excitations

According to the proofing in Section 2.1, it is clear that the weak energy–momentum method can be applied to a class of stochastic systems with positive or negative definite potential energy since whether the potential energy is positive or negative definite is not a prerequisite or restrictive condition during the proving procedure. Considering the existing methods used for stochastic stability analysis of a stochastic gyroscopic system with positive definite potential energy are difficult to employ in studying stochastic stability of a stochastic gyroscopic system with negative definite potential energy, we will aim at a stochastic gyroscopic system with negative definite potential energy in the following studies, and the corresponding expressions with regard to a stochastic gyroscopic system with positive definite potential energy can be obtained analogically.

Consider a stochastic gyroscopic system with negative definite potential energy. The Lagrangian equations of motion of the gyroscopic system with $n$ (even positive integer) degree of freedom is of the form

$$m_{ii}\ddot{q}_i + 2g_{ij}\dot{q}_j + d_{ii}\dot{q}_i + (k_{ii} - n_{ii})q_i = \varepsilon s_{ii}\xi(t)q_i, \quad i,j = 1,2,...,n. \tag{1}$$

where $\boldsymbol{q} = \begin{Bmatrix} q_1 & q_2 & \dots & q_n \end{Bmatrix}^{\mathrm{T}}$ is the generalized displacement vector. $\boldsymbol{M} = \mathrm{diag}(m_{ii})$ is the constant and positive definite matrix of mass or moment of inertia. $\boldsymbol{G} = [g_{ij}]$ is the constant and anti-symmetric gyroscopic matrix. $\boldsymbol{D} = \mathrm{diag}(d_{ii})$ is the damping matrix caused by dissipative forces. $\boldsymbol{K}_k = \mathrm{diag}(k_{ii})$, $\boldsymbol{K}_n = \mathrm{diag}(n_{ii})$ and $\boldsymbol{K}_s = \mathrm{diag}(s_{ii})$ are the stiffness matrices related to different items