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Stability of two-time scale linear stochastic hybrid systems

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ABSTRACT

The problem of exponential mean-square stability of two-time scale, linear stochastic hybrid systems has been studied in this paper. To obtain the sufficient conditions of stability, two basic approaches of stability analysis: for one-time hybrid systems with a Markovian switching rule as well as switching rule and singularly perturbed nonhybrid systems, were combined. The Lyapunov techniques were used in both approaches.

The determination of mathematical model of a real dynamic system is the fundamental problem of engineering research. In many cases the classical models are not appropriate and researchers are looking for more complex models. Among them recently two groups of models are considered, namely multi-scale and hybrid models.

In the case of multi-scale models one of the most important groups is the so-called multi-time scale models. They are used in the case when in some real dynamic systems one can observe an interaction of a few processes acting with different speeds, for instance, if slow and fast vibrations appear in dynamic system. The dynamic of these systems is described by the so-called singularly perturbed or two time scale models. A wide study of such systems is given for instance in [6].

Another group of complex models is variable structure or hybrid models described by deterministic or stochastic differential [2,4,5]. In the successive moments of time their structures can change according to the given switching rule thereupon creates the hybrid system. Since systems in the real world often need to run for a long period of time, the stability and control of hybrid systems have recently received a lot of attention (see for example [5]). Hybrid systems are applicable in many fields including areas of nuclear, thermal, chemical processes, biology, socioeconomics, immunology and many others. Also the theory of hybrid systems is used in control of mechanical systems, for instance nonsmooth or nonholonomic systems and many other fields. The problem of stability of stochastic hybrid systems was very intensive studied in the literature, see for instance [5,8]. This problem is very important, because it is a well known fact that even if all structures are stable the whole hybrid system with a special switching signal can be unstable [5].

Some authors tried to combine both approaches. They have studied control and stabilization problems of linear singularly perturbed deterministic hybrid systems with a Markovian switching rule [10,1,11]. The objective of this paper is to extend this approach to the study of the stability of linear singularly perturbed stochastic hybrid systems with parametric excitation and Markovian switching rule or other switching rules, using stability analysis of singularly perturbed deterministic nonhybrid systems [3] and its extended version for stochastic nonhybrid systems [9].

1. Mathematical preliminaries

Throughout this paper we use the following notation. Let $|\cdot|$ and $\langle\cdot\rangle$ be the Euclidean norm and the inner product in $\mathbb{R}^n$, respectively. By $\lambda(\mathbf{A})$ we denote the eigenvalue of the matrix $\mathbf{A}$, $\lambda_{min}(\mathbf{A})$ and $\lambda_{max}(\mathbf{A})$ denote the smallest and the biggest real part of the eigenvalue of the matrix $\mathbf{A}$, respectively. We will denote the indicator function of a set G by $\mathbb{I}_G$. We mark $\mathbb{R}_+=[0,\infty)$, $\mathbb{T}=[t_0,\infty)$, $t_0\geq 0$. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying usual conditions. Let $\xi(t)=[\xi_1(t),\ldots,\xi_m(t)]^T,\ t\geq 0$, be the m-dimensional Wiener process defined on the probability space. The process $r(t),\ t\geq 0$, be a right-continuous switching signal (cadlag) (Markov chain) on the probability space taking values in a finite state space $\mathbb{S}=\{1,2,\ldots,N\}$

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with a generator $\Gamma = [\gamma_{ij}]_{N\times N}$, i.e.

$$\mathbb{P}\{r(t+\delta)=j|r(t)=i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\ 1+\gamma_{ii}\delta + o(\delta) & \text{if } i = j, \end{cases} \quad (1)$$

where $\delta > 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, $\gamma_{ii} = -\sum_{i\neq j}\gamma_{ij}$. We assume that the Markov chain is irreducible, i.e. rank$(\Gamma) = N-1$, and has a unique stationary distribution $\mathcal{P} = [p_1, p_2, \ldots, p_N]^T \in \mathbb{R}^N$ which can be determined by solving

$$\begin{cases} \mathcal{P}\Gamma = \mathbf{0} \\ \text{subject to } \sum_{i=1}^{N} p_i = 1 \quad \text{and} \quad p_i > 0 \text{ for all } i \in \mathbb{S}. \end{cases} \quad (2)$$

We consider the nonlinear hybrid system with multiplicative excitations described by the vector Itô differential equation

$$d\mathbf{X}(t) = \mathbf{\Phi}(\mathbf{X}, t, \sigma(t))\,dt + \sum_{k=1}^{m} \mathbf{G}_k(\mathbf{X}, t, \sigma(t))\,d\xi_k(t), \quad \mathbf{X}(0) = \mathbf{X}_0, \quad (3)$$

$t \geq 0$, $\sigma(0) = \sigma_0 \in \mathbb{S}$, $\mathbf{X} \in \mathbb{R}^n$, $\mathbf{\Phi}, \mathbf{G}_k : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^n$, $k=1,\ldots,m$, $\mathbf{X} = [X_1, \ldots, X_n]^T$, $\mathbf{\Phi} = [\Phi^1, \ldots, \Phi^n]^T$, $\mathbf{G}_k = [G_k^1, \ldots, G_k^n]^T$, the process $\sigma(t) : \mathbb{R}_+ \to \mathbb{S}$ is a switching signal. We assume that the solution $\mathbf{X}(t)$ of Eq. (3) exists and is everywhere continuous with probability one.

Additionally, we assume that processes w_k, the process $\sigma(t)$ and the initial condition are mutually independent and processes $w_k(t)$ and $\sigma(t)$ are $\{\mathcal{F}_t\}_{t\geq 0}$ adapted.

A few cases of switching signals are usually considered in the literature. In this paper we will study two of them, namely

(i) any piecewise constant deterministic function,
(ii) Markov switching rule $\sigma(t) = r(t)$.

We quote some results that will be useful in further considerations [7], [8].

For any function $V(\mathbf{X}, t, l)$ twice differentiable in $\mathbf{X}$ and once differentiable in t the l-th process defined by Eq. (3) has a generator $\mathcal{L}_l$ given by

$$\begin{aligned} \mathcal{L}_l^{(3)} V(\mathbf{X}, t, l) = {} & \frac{\partial V(\mathbf{X}, t, l)}{\partial t} + \sum_{i=1}^{n} \Phi^i(\mathbf{X}, t, l)\frac{\partial V(\mathbf{X}, t, l)}{\partial X_i} \\ & + \frac{1}{2}\sum_{r,s=1}^{n}\sum_{k=1}^{m} G_k^r(\mathbf{X}, t, l) G_k^s(\mathbf{X}, t, l)\frac{\partial^2 V(\mathbf{X}, t, l)}{\partial X_r \partial X_s} \\ & + \sum_{j=1}^{N} \gamma_{lj} V(\mathbf{X}, t, l), \quad l \in \mathbb{S}. \end{aligned} \quad (4)$$

When the switching rule $\sigma(t)$ is not equal to Markov switching signal $r(t)$ then in equality (4) it is assumed that $\gamma_{lj} = 0$.

Definition 1. The null solution $\mathbf{X}(t) \equiv 0$ of the stochastic differential equation (3) is said to be p-th mean exponentially stable $(p > 0)$ if there exists a pair of positive scalars α, c such that $\forall (\mathbf{X}_0, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$

$$\mathbb{E}[|\mathbf{X}(\mathbf{X}_0, t_0)|^p] \leq c\mathbb{E}[|\mathbf{X}_0|^p]\exp\{-\alpha(t-t_0)\}, \quad t \in \mathbb{T} \quad (5)$$

Theorem 1 (*Mao [8]*). *Assume that there exists a function $V(\mathbf{X}, t, l)$ twice differentiable in $\mathbf{X}$ and once differentiable in t for all $l \in \mathbb{S}$ and positive numbers p, λ, c_1 and c_2 such that*

$$c_1|\mathbf{X}|^p \leq V(\mathbf{X}, t, l) \leq c_2|\mathbf{X}|^p, \quad l \in \mathbb{S} \quad (6)$$

$$\mathcal{L}_l^{(3)} V(\mathbf{X}, t, l) \leq -\lambda|\mathbf{X}|^p, \quad l \in \mathbb{S} \quad (7)$$

then the null solution $\mathbf{X}(t)$ of Eq. (3) for $\sigma(t) = r(t)$ is p-th mean exponentially stable.

Corollary 1. *If assumption (7) is replaced by*

$$\mathcal{L}_l^{(3)} V(\mathbf{X}, t, l) \leq -2\alpha V(\mathbf{X}, t, l), \quad l \in \mathbb{S} \quad (8)$$

for a positive parameter α, then the null solution $\mathbf{X}(t)$ of Eq. (3) is also p-th mean exponentially stable.

Lemma 1 (*Corless and Glielmo [3]*). *Consider any symmetric matrix $S(\varepsilon) = [s_{ij}(\varepsilon)]$, $i, j = 1, 2$, in which the function $s_{ij} : (0, +\infty) \to R$ satisfy*

$$\lim_{\varepsilon\to 0} s_{11}(\varepsilon) = \lambda_0, \quad \lim_{\varepsilon\to 0} s_{22}(\varepsilon) = +\infty, \quad \lim_{\varepsilon\to 0}\frac{s_{12}^2(\varepsilon)}{s_{22}(\varepsilon)} = 0, \quad (9)$$

then

$$\lim_{\varepsilon\to 0}\alpha_{min}(S(\varepsilon)) = \lambda_0, \quad (10)$$

where $\alpha_{min}(S)$ is the minimal eigenvalue of matrix S, λ_0 is a constant.

2. Problem formulation

We consider a singularly perturbed stochastic linear hybrid system with the Markov switching rule $\sigma(t) = r(t)$ described by the vector Ito differential equations

$$\begin{aligned} d\mathbf{x}(t) = {} & [\mathbf{A}_1(r(t))\mathbf{x} + \mathbf{B}_1(r(t))\mathbf{z}]\,dt \\ & + \left[\sum_{k=1}^{M} \mathbf{G}_k^1(r(t))\mathbf{x} + \sum_{k=1}^{M} \mathbf{C}_k^1(r(t))\mathbf{z}\right] d\xi_k^1(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \end{aligned} \quad (11)$$

$$\begin{aligned} \varepsilon\, d\mathbf{z}(t) = {} & [\mathbf{A}_2(r(t))\mathbf{x} + \mathbf{B}_2(r(t))\mathbf{z}]\,dt \\ & + \sqrt{\varepsilon}\left[\sum_{k=1}^{M} \mathbf{G}_k^2(r(t))\mathbf{x} + \sum_{k=1}^{M} \mathbf{C}_k^2(r(t))\mathbf{z}\right] d\xi_k^2(t), \quad \mathbf{z}(t_0) = \mathbf{z}_0, \end{aligned} \quad (12)$$

where $t \in \mathbb{R}_+$ is the time, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{z} \in \mathbb{R}^m$ are the state vectors and $\varepsilon > 0$ is the singular perturbation parameter. For some $\varepsilon^* > 0$ the matrices $\mathbf{A}_1, \mathbf{G}_k^1$ are $n \times n$, $\mathbf{B}_1, \mathbf{C}_k^1$ are $n \times m$, $\mathbf{A}_2, \mathbf{G}_k^2$ are $m \times n$ and $\mathbf{B}_2, \mathbf{C}_k^2$ are $m \times m$ dimensional; $\xi_k^1(t)$ and $\xi_k^2(t)$, $k = 1, \ldots, M$, are independent standard Wiener processes. For convenience we assume that the initial conditions $\mathbf{x}_0 \in \mathbb{R}^n$ and $\mathbf{z}_0 \in \mathbb{R}^m$ are deterministic.

For each $\mathbf{x} \in \mathbb{R}^n$ and $l \in \mathbb{S}$ we assume that the equation $\mathbf{A}_2(r(t))\mathbf{x} + \mathbf{B}_2(r(t)\mathbf{z} = 0$ has a unique solution

$$\mathbf{z} = \mathbf{h}(\mathbf{x}) = -\mathbf{B}_2^{-1}(r(t))\mathbf{A}_2(r(t)\mathbf{x}. \quad (13)$$

This assumption defines the complete reduced-order system by setting $\varepsilon = 0$ and $\mathbf{z} = \mathbf{h}(\mathbf{x})$ in (11) as follows:

$$\begin{aligned} d\mathbf{x}(t) = {} & [\mathbf{A}_1(r(t)) - \mathbf{B}_1(r(t))\mathbf{B}_2^{-1}(r(t))\mathbf{A}_2(r(t))]\mathbf{x}\,dt \\ & + \left[\sum_{k=1}^{M} \mathbf{G}_k^1(r(t)) - \mathbf{C}_k^1(r(t))\mathbf{B}_2^{-1}(r(t))\mathbf{A}_2(r(t))\mathbf{x}\right] d\xi_k^1(t). \end{aligned} \quad (14)$$

For system (11) and (12) we define a new variable $\tau = (t - t_0)/\varepsilon$ and new τ-dependent variables $\mathbf{x}_f(\tau) = \mathbf{x}(t_0 + \varepsilon\tau) = \mathbf{x}(t)$, $\mathbf{z}_f(\tau) = \mathbf{z}(t_0 + \varepsilon\tau) = \mathbf{z}(t)$, $r_f(\tau) = r(t_0 + \varepsilon\tau) = r(t)$, $w_k^1(\tau) = \sqrt{\varepsilon}\xi_k^1(t_0 + \varepsilon\tau)$, $w_k^2(\tau) = \sqrt{\varepsilon}\xi_k^2(t_0 + \varepsilon\tau)$. The stochastic differential equations for these new variables have the form

$$\begin{aligned} d\mathbf{x}_f(\tau) = {} & \varepsilon[\mathbf{A}_1(r_f(\tau))\mathbf{x}_f + \mathbf{B}_1(r_f(\tau))\mathbf{z}_f]\,d\tau \\ & + \sqrt{\varepsilon}\left[\sum_{k=1}^{M} \mathbf{G}_k^1(r_f(\tau))\mathbf{x}_f + \sum_{k=1}^{M} \mathbf{C}_k^1(r_f(\tau))\mathbf{z}_f\right] dw_k^1(\tau), \end{aligned} \quad (15)$$

$$\begin{aligned} d\mathbf{z}_f(\tau) = {} & [\mathbf{A}_2(r_f(\tau))\mathbf{x}_f + \mathbf{B}_2(r_f(\tau))\mathbf{z}_f]\,d\tau \\ & + \left[\sum_{k=1}^{M} \mathbf{G}_k^2(r_f(\tau))\mathbf{x}_f + \sum_{k=1}^{M} \mathbf{C}_k^2(r_f(\tau))\mathbf{z}_f\right] dw_k^2(\tau). \end{aligned} \quad (16)$$

For $\varepsilon = 0$ Eq. (15) becomes $d\mathbf{x}_f(\tau) = \mathbf{0}$, which implies that $\mathbf{x}_f(\tau) = \mathbf{x}_f(0) = \mathbf{x}(t_0) = \mathbf{x}_0$. Thus the boundary-layer system is described by

$$\begin{aligned} d\mathbf{z}_f(t) = {} & [\mathbf{A}_2(r_f(\tau))\mathbf{x}_0 + \mathbf{B}_2(r_f(\tau))\mathbf{z}_f]\,dt \\ & + \left[\sum_{k=1}^{M} \mathbf{G}_k^2(r_f(\tau))\mathbf{x}_0 + \sum_{k=1}^{M} \mathbf{C}_k^2(r_f(\tau))\mathbf{z}_f\right] dw_k^2(t). \end{aligned} \quad (17)$$

We note that $\mathbf{x}_0$ is treated in Eq. (17) as a vector of parameters.