Mechanism and Machine Theory 76 (2014) 70–85

Contents lists available at ScienceDirect

Mechanism and Machine Theory

journal homepage: www.elsevier.com/locate/mechmt

Higher derivatives of the kinematic mapping and some applications

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A R T I C L E   I N F O

Article history:
Received 6 November 2012
Received in revised form 3 January 2014
Accepted 9 January 2014
Available online 22 March 2014

Keywords:
Mechanisms
Screw systems
Higher-order kinematics
Lie groups
Motion groups
Mobility

A B S T R A C T

A thorough analysis of mechanisms requires higher derivatives of the kinematic relations between its members. Such a relation is the (forward) kinematic mapping of a kinematic chain that relates the joint motions to the motion of the links. A first-order motion approximation is provided by the instantaneous joint screws. Higher-order approximations thus require higher derivatives of this screw system. Since the representation of screw entities is not unique a particular representation gives rise to a particular explicit form of the derivatives. Two commonly used variants are the spatial and body-fixed representation.

Here a closed form expression for the ν-th partial derivatives of the joint screws within a kinematic chain w.r.t., the joint variables is presented for the spatial and body-fixed representation. The form of the final expressions makes them ideal for computer implementations. The expressions are given explicitly for derivatives of up to 4th order.

The paper concludes with a brief discussion of applications where higher derivatives are relevant. These are the kinematic analysis and determination of motion spaces of serial mechanisms, the higher-order mobility analysis, and the algebraic formulation of motion equations.

1. Introduction

A kinematic mapping relates the motion of a rigid body to the joint motions of a kinematic chain. Its time derivatives yield the twist, acceleration, jerk, etc. of the body. Time derivatives of the twists of members in a kinematic chain and derivatives of screws are essential operations in kinematics [2,9,26]. It may be surprising that this problem has been the topic of relatively recent publications. For instance Bruynickx and De Schutter [11] presented explicit expressions for the first derivative of the joint screws in a kinematic chain and the relations between body-fixed, spatial, and hybrid twist representations. A good overview on this topic can be found in [32,69]. Recognizing the Lie group nature of rigid body motions, and correspondingly the Lie algebra nature of screws, Karger [30], Rico et al. [55], and Lerbet [31] derived closed form expressions of higher time derivatives. Using the Lie group $SE(3)$ the treatment of kinematic chains has rapidly developed documented by the books by Murray et al. [50] and Selig [60].

Besides the higher-order kinematic analysis of mechanisms the efficient determination of derivatives of twists is also vital for analyzing the dynamic behavior and stability of multibody systems (MBS). The crucial observation is that the twist of the terminal body in a kinematic chain can be recursively determined by propagating the twists of the bodies in the chain, and consequently that there is a recursive relation for the partial derivatives of the kinematic mapping. This fact was exploited in the context of multibody dynamics simulation from its outset. It gave rise to dynamics formulations that have a complexity linear in the number of bodies, originally proposed in [15] and advanced in [1,5,6,11,12,16,17,57,58,39,63], only to mention a few key publications. Higher-order derivatives of kinematic quantities are required for linearization of motion equations and for sensitivity analysis as pursued in [25]. The formulations in all these publications are derived upon standard vector algebra using the hybrid representation of body twists. Later the problem was approached with the theory of screws and Lie groups resulting in several publications such as [10,38,42,50,52,53]. The advantage of these Lie group methods is that the resulting relations are very

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http://dx.doi.org/10.1016/j.mechmachtheory.2014.01.007

compact, and that the method is conceptually simple. The Lie group setting will also be used in this paper, and it will be shown that therewith higher partial derivatives can be derived in a systematic and compact manner.

The aim of this paper is to provide explicit expressions for the partial derivatives of a kinematic mapping, and to point out its relevance for selected problems in mechanisms and robotics. To the author's knowledge the explicit expressions for higher order derivatives in spatial and body-fixed representation have not been presented elsewhere. The publication that is probably closest to this paper is [13] where derivatives of the spatial representation up to fourth order were reported.

In Section 2 the screw description of a kinematic chain is recalled. Two different formulations are introduced, one making use of local reference frames while in the other formulation all screws and rigid body transformations are described in a single spatial reference frame. The spatial and body-fixed twist of a rigid body are recalled arising from right and left trivialization of the Lie group of rigid body motions, respectively, and are expressed in terms of the instantaneous joint screws. The latter are the basis for the expressions of the derivatives in Section 3. In this section explicit expressions for the partial derivatives of the instantaneous joint screws w.r.t. to the joint variables for any order are presented for the spatial as well as the body-fixed twist. Section 4 discusses some implications and application where the presented expressions are relevant. An obvious application is the kinematic analysis of a serial kinematic chain, for which the presented results give rise very compact expressions. Another important application is the higher-order mobility analysis, which is based on the compatibility equations for velocity, acceleration, jerk, etc. As a direct consequence of the involved equations it is briefly noticed that the motion space of a kinematic chain is determined by nested Lie brackets of the joint screws. It is further pointed out that the expressions for the derivatives allow for statements on how a serial manipulator can escape from a singularity. Finally it is recalled that the algebraic form of the partial derivatives is relevant for generating motion equations of multibody systems.

Throughout the paper the established notation for screws and the Lie group of rigid body motions are used. $\mathbf{X} = (\omega, \mathbf{v}) \in \mathbb{R}^3 \times \mathbb{R}^3$ denotes the screw coordinate vector of a screw when expressed in a given frame. If the screw represents a twist, it is denoted with $\mathbf{V}$. The matrix representation of the Lie group of rigid body motions, $SE(3)$, with Lie algebra $se(3)$ is used. To a screw coordinate vector $\mathbf{X}$ is assigned an $se(3)$ matrix as $\widehat{\mathbf{X}} = \begin{pmatrix} \widehat{\omega} & \mathbf{v} \\ 0 & 0 \end{pmatrix} \in se(3)$, where $\widehat{\omega} \in so(3)$ is the skew symmetric matrix associated to vector $\boldsymbol{\omega}$ so that $\boldsymbol{\omega} \times \mathbf{x} = \widehat{\omega}\mathbf{x}$. This isomorphism allows to identify the $se(3)$ Lie bracket with the screw product via

$$\widehat{\mathbf{Y}} = [\widehat{\mathbf{X}}_1, \widehat{\mathbf{X}}_2] = \widehat{\mathbf{X}}_1\widehat{\mathbf{X}}_2 - \widehat{\mathbf{X}}_2\widehat{\mathbf{X}}_1 \tag{1}$$

since $\mathbf{Y} = [\mathbf{X}_1, \mathbf{X}_2] = (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2, \boldsymbol{\omega}_1 \times \mathbf{v}_2 + \mathbf{v}_1 \times \boldsymbol{\omega}_2)$. $\mathbf{C} \in SE(3)$ determines a frame transformation representing a rigid body motion, i.e. finite screw motion. A transformation of screw coordinates is described by the adjoint operator: $\widehat{\mathbf{Y}} = \mathrm{Ad}_{\mathbf{C}}(\widehat{\mathbf{X}}) = \mathbf{C}\widehat{\mathbf{X}}\mathbf{C}^{-1}$, which is in vector representation of screws determined by $\mathbf{Y} = \mathbf{Ad_C X}$ with 6×6 matrix $\mathbf{Ad_C}$. The exponential mapping $\mathbf{C}(t) = \exp(\widehat{\mathbf{X}}(t))$ gives the finite motion $\mathbf{C}(t) \in SE(3)$ corresponding to a continuous evolution of its associated instantaneous screw $\widehat{\mathbf{X}}(t) \in se(3)$. For simplicity the hat symbol over the argument $\mathbf{X}$ is often omitted. Details on the Lie group background can be found in [50,60]. The reader not familiar with the Lie group method can simply view it as another way of dealing with homogeneous transformation matrices allowing for canonical parameterizations.

2. Kinematics of a serial chain

2.1. Configurations and the kinematic mapping

Consider a kinematic chain comprising n rigid bodies numbered consecutively with $r = 1,\ldots, n$ from the ground to the terminal body. W.l.o.g. all joints are assumed to have 1 DOF, i.e. lower kinematic pairs are modeled as conglomerate of revolute, prismatic, or screw joints. This is equivalent to the introduction of canonical coordinates of second kind on the motion groups associated to the joints. Joints are numbered with the same indices as bodies. Joint r connects body $r - 1$ with body r. To each joint can be assigned a screw coordinate vector. While screw theory allows for coordinate-free treatment of kinematics, the actual computation requires introduction of reference frames. There are two sensible choices of reference frame, namely body-fixed and space-fixed (inertial).

In matrix representation of $SE(3)$ the configuration of body r is expressed in form of the 4×4 matrix

$$\mathbf{C}_r = \begin{pmatrix} \mathbf{R}_r & \mathbf{p}_r \\ 0 & 1 \end{pmatrix}$$

that transforms homogenous point coordinates. $\mathbf{R}_r \in SO(3)$ is the rotation matrix that determines the rotation of the body-fixed to space-fixed frame, and $\mathbf{p}_r \in \mathbb{R}^3$ the position vector from the origin of the space fixed to the body-fixed frame expressed in the space-fixed frame.

2.1.1. Body-fixed reference frames

To each body is attached a reference frame (RFR), which kinematically represents the body as shown in Fig. 1. A body-fixed description of the relative kinematics of two bodies requires splitting the relative configuration into a constant and a variable part.