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Linear models for non-Gaussian processes and applications to linear random vibration

Mircea Grigoriu *

Cornell University, 363 Hollister Hall, Ithaca, NY 14853, USA

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ABSTRACT

Linear models are finite sums of specified deterministic, continuous functions of time with random coefficients. It is shown that linear models provide (*i*) accurate approximations for real-valued non-Gaussian processes with continuous samples defined on bounded time intervals, (*ii*) simple solutions for linear random vibration problems with non-Gaussian input, and (*iii*) efficient techniques for selecting optimal designs from collections of proposed alternatives. Theoretical arguments and numerical examples are presented to establish properties of linear models, illustrate the construction of linear models, solve linear random vibration with non-Gaussian input, and propose an approach for optimal design of linear dynamic systems. It is shown that the proposed linear model provides an efficient tool for analyzing linear systems in non-Gaussian environment.

1. Introduction

It is common in applications to examine the performance of proposed alternative designs for dynamic systems subjected to random input processes and select designs that are optimal in some sense. For example, wind tunnel studies and response estimates are used to extract final designs for high rise buildings from proposed alternatives. The calculation of response statistics poses notable difficulties since wind loads are non-Gaussian processes [1], and there exists no simple, accurate, and efficient method for calculating statistics of even linear systems in non-Gaussian environment. Monte Carlo simulation is the only general method for solving linear random vibration with non-Gaussian noise but its computation demand can be excessive. This limits the number of alternative designs that can be investigated and/or the number of samples used to estimate response statistics.

We develop linear models for non-Gaussian stationary and nonstationary stochastic processes, that are defined by finite sums of specified deterministic function with random coefficients. For example, let $X(t)$, $t \in [0, \tau]$, be a real-valued stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$, and denote by

$$X^{(n)}(t, \omega) = C_0(\omega) + \sum_{k=1}^{n} C_k(\omega)\, \theta_k(t), \quad t \in [0, \tau],\ n = 1, 2, \ldots, \tag{1}$$

a sequence of linear models for $X(t)$, where $\{C_k\}$ are random variables defined on $(\Omega, \mathcal{F}, P)$ and $\{\theta_k\}$ are specified, deterministic functions of time.

Linear models have been and are used extensively in applications to represent approximately the first two moments for stochastic processes $X(t)$ with finite variance and the probability law for Gaussian processes. For example, if $X(t)$ is a weakly stationary m.s. continuous process with mean 0 and one-sided spectral density $g(\nu)$, it admits the spectral representation

$$X(t) = \int_0^\infty (\cos(\nu t)\, \mathrm{d}U(\nu) + \sin(\nu t)\, \mathrm{d}V(\nu)), \tag{2}$$

where the integral is defined in the mean square sense and U and V are real-valued processes with orthogonal increments such that $E[U(\nu)] = E[V(\nu)] = 0$, $E[\mathrm{d}U(\nu)\, \mathrm{d}V(\nu')] = 0$, and $E[\mathrm{d}U(\nu)^2] = E[\mathrm{d}V(\nu)^2] = g(\nu)\, \mathrm{d}\nu$ ([2], Section 3.9.4.1). If $X(t)$ is or is not weakly stationary, has mean 0, and its correlation function is continuous and square integrable in $[0, \tau] \times [0, \tau]$, then

$$X(t, \omega) = \sum_{k=1}^{\infty} \lambda_k^{1/2}\, \theta_k(t)\, Y_k(\omega), \quad t \in [0, \tau], \tag{3}$$

where $\{Y_k\}$ are uncorrelated random variables with mean 0 and variance 1 and $\{\lambda_k, \theta_k(t)\}$ denote the eigenvalues and eigenfunctions of the operator equation $\mathcal{A}[\theta] = \lambda\theta$ with $\mathcal{A}[\theta(t)] = \int_0^\tau E[X(s)\, X(t)]\, \theta(s)\, \mathrm{d}s$. The series in Eq. (3), referred to as the Karhunen–Loéve expansion of $X(t)$, is convergent in the mean square sense ([3], Section 6.2). Discrete versions of the spectral representation in Eq. (2) and truncated versions of

* Tel.: +1 607 255 3334.
E-mail address: mdg12@cornell.edu.

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the Karhunen–Loéve expansion in Eq. (3) define linear models of the type in Eq. (1). These models provide no information beyond the first two moments of $X(t)$, unless this process is Gaussian. Relationships between spectral representation and Karhunen–Loéve expansions are discussed in [4].

Linear models provide a conceptually simple and computationally efficient solutions to linear random vibration problems with non-Gaussian input. Let $U(t)$ be the solution of a linear random vibration problem defined by a stochastic differential equation $\mathcal{L}[U(t)] = X(t)$, where $\mathcal{L}$ is a linear operator. The mean and covariance functions of $U(t)$ can be calculated simply by methods of linear random vibration ([2], Section 7.2.1.1). Monte Carlo simulation is the only general method for finding higher order properties of $U(t)$. We propose to approximate $U(t)$ by the solution $U^{(n)}(t)$ of $\mathcal{L}[U^{(n)}(t)] = X^{(n)}(t)$, where $X^{(n)}(t)$ is a linear model of $X(t)$. Since $\mathcal{L}$ is a linear operator, $U^{(n)}(t) = \sum_{k=0}^{n} C_k\, \mathcal{L}^{-1}[\theta_k(t)]$ is a linear model for $U(t)$, where $\mathcal{L}^{-1}[\theta_k(t)]$ are responses of the dynamic system to the deterministic functions $\{\theta_k(t)\}$ and $\theta_0(t) = 1$. Statistics of $U^{(n)}(t)$ can be calculated simply and efficiently from its expression and properties of the random variables $\{C_k\}$.

Our objectives are to (1) develop a method for constructing linear models $X^{(n)}(t)$ for non-Gaussian processes $X(t)$, (2) assess the accuracy of linear models as a function of their order n and sample properties of $X(t)$, (3) solve linear random vibration problems with non-Gaussian input by linear models, and (4) show that the proposed linear model provides an accurate tool for analyzing linear systems in non-Gaussian environment that can be used to assess the performance of a large number of alternative designs. Theoretical arguments are used to prove that under some conditions the processes $X^{(n)}(t)$ and $X(t)$ as well as the solutions $U^{(n)}(t)$ and $U(t)$ of linear systems subjected to these processes have similar statistics. These arguments are supported by numerical examples.

2. Trigonometric polynomials

Let $x(t)$ be a real-valued, continuous, periodic function with period $\tau > 0$. We set $\tau = 2\pi$ without loss of generality since $y(\tilde{t}) = x(\tilde{t}\,\tau/(2\pi))$, $\tilde{t} \in [0, 2\pi]$, obtained from $x(t)$ by distorting its time scale is a real-valued, continuous, periodic function with period 2π. Accordingly, $x(t)$ is considered to be a member of the set $C_{\text{per}}[0, 2\pi]$ of real-valued, continuous, periodic functions with period 2π, so that $x(t) = x(t + 2\,r\,\pi)$, $r \in \mathbb{Z}$, at all times $t \in \mathbb{R}$.

Let

$$\mathcal{P}_n[0, 2\pi] = \left\{ q(t) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k\, \cos(k\,t) + b_k\, \sin(k\,t)] \right\} \tag{4}$$

denote the space of trigonometric polynomials of degree or order n, where $\{a_k\}$ and $\{b_k\}$ are real-valued coefficients. For a fixed n, $\mathcal{P}_n[0, 2\pi]$ is a linear subspace of $C_{\text{per}}[0, 2\pi]$. Following are the properties of $\mathcal{P}_n[0, 2\pi]$ that are relevant to our objective of constructing linear models for non-Gaussian stochastic processes.

1. *For any $x \in C_{\text{per}}[0, 2\pi]$ and $\varepsilon > 0$, there exists $q \in \mathcal{P}_n[0, 2\pi]$ such that* ([5], Theorem 13.1)

$$\|x - q\|_\infty = \sup_{0 \le t \le 2\pi} |x(t) - q(t)| \le \varepsilon. \tag{5}$$

This theorem shows that it is always possible to find a degree n and a polynomial $q \in \mathcal{P}_n[0, 2\pi]$ that is as close as desired to $x \in C_{\text{per}}[0, 2\pi]$. Hence, optimization algorithms searching for a pair (n, q) with the above properties can be constructed.

2. *The trigonometric polynomials in $\mathcal{P}_n[0, 2\pi]$ minimize the least square distance function $d(x, q) = [\int_{-\pi}^{\pi} (x(t) - q(t))^2\, \mathrm{d}t]^{1/2}$ if and only if the coefficients $\{a_k, b_k\}$ of $q \in \mathcal{P}_n[0, 2\pi]$ are the Fourier coefficients of $x(t)$ given by* ([5], Theorem 13.2)

$$a_k = \frac{1}{\pi} \int_0^{2\pi} x(t)\, \cos(k\,t)\, \mathrm{d}t, \quad k = 0, 1, \ldots, n,$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} x(t)\, \sin(k\,t)\, \mathrm{d}t, \quad k = 1, 2, \ldots, n. \tag{6}$$

The resulting trigonometric polynomial q is a truncated Fourier series of $x(t)$. Since the mean square error of q is ([6], Section 2.5)

$$\int_0^{2\pi} (x(t) - q(t))^2\, \mathrm{d}t = \int_0^{2\pi} x(t)^2\, \mathrm{d}t - \left[\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right], \tag{7}$$

it is possible to find a truncation level n for the Fourier series of $x(t)$ such that its m.s. error does not exceed a specified value.

3. *If $x \in C_{\text{per}}^{(1)}[0, 2\pi]$, then* ([5], Theorem 15.1)

$$\min_{q \in \mathcal{P}_n[0, 2\pi]} \|x - q\|_\infty \le \frac{\pi}{2(n+1)}\, \|\dot{x}\|_\infty, \tag{8}$$

where $C_{\text{per}}^{(r)}[0, 2\pi]$, $r = 1, 2, \ldots$, is the linear space of real-valued periodic functions with period 2π that have continuous rth derivative and $\dot{x}(t) = dx(t)/\mathrm{d}t$.

The bound in Eq. (8) shows that the error of the optimal member of $\mathcal{P}_n[0, 2\pi]$, that is, $q \in \mathcal{P}_n[0, 2\pi]$ with the smallest error $\|x - q\|_\infty$, is of order $O(n^{-1})$, and that it is possible to select a polynomial q^* of order n^* such that $\|x - q^*\|_\infty$ does not exceed a specified value.

We focus on linear models of the type in Eq. (1) with $\{\theta_k(t)\}$ selected from the complete set of functions $(1, \cos(t), \sin(t), \ldots, \cos(k\,t), \sin(k\,t), \ldots)$ spanning $C_{\text{per}}[0, 2\pi]$, so that the linear models under consideration are members of $\mathcal{P}_n[0, 2\pi]$. The emphasis is on these types of models since trigonometric polynomials have been and are used extensively to construct approximate representations for stochastic processes. Algebraic and other polynomials can be used for the basis functions $\{\theta_k(t)\}$ since they have properties similar to those of the members of $\mathcal{P}_n[0, 2\pi]$ ([5], Chapters 12 and 16).

If $x(t)$ is periodic with period 2π but $x(0) \neq x(2\pi)$, there is no sequence of polynomials in $\mathcal{P}_n[0, 2\pi]$ that converges to $x(t)$ at $t = 2\,r\,\pi$, $r \in \mathbb{Z}$ ([6], Section 1.10). However, $x(t)$ can be altered such that its modified version is in $C_{\text{per}}[0, 2\pi]$ and its values at the ends of $[0, 2\pi]$ coincide. Let

$$y(\tilde{t}) = x(\tilde{t})\, 1(0 \le \tilde{t} \le 2\pi - \varepsilon) + [x(2\pi) + x(0) - x(2\pi)\, (\tilde{t} - 2\pi + \varepsilon)/\varepsilon] 1(2\pi - \varepsilon < \tilde{t} \le 2\pi), \tag{9}$$

where $\varepsilon \in (0, 2\pi)$ is arbitrary and $\tilde{t} = (1 - \varepsilon/(2\pi))\, t$, $t \in [0, 2\pi]$. We note that $y \in C_{\text{per}}[0, 2\pi]$, $y(0) = y(2\pi)$, $y(\tilde{t})$ for $\tilde{t} \in [0, 2\pi - \varepsilon]$ gives $x(t)$ for $t \in [0, 2\pi]$ viewed in a distorted clock with time units $\tilde{t}$, and $y(\tilde{t})$ in $[2\pi - \varepsilon, 2\pi]$ is a line connecting the points $y(2\pi - \varepsilon) = x(2\pi)$ and $y(2\pi) = x(0)$. Higher order splines can be used to define $y(\tilde{t})$ in $[2\pi - \varepsilon, 2\pi]$ such that, for example, $y \in C_{\text{per}}^{(1)}[0, 2\pi]$ if $x(t)$ is continuously differentiable. Once a polynomial representation $q(\tilde{t})$ has been developed for $y(\tilde{t})$, $x(t)$, $t \in [0, 2\pi]$, can be approximated by $q(2\pi\,\tilde{t}/(2\pi - \varepsilon))$, $\tilde{t} \in [0, 2\pi - \varepsilon]$. We also note that these considerations extend directly to functions $x(t)$ that are not periodic but are defined on a bounded interval.