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Probabilistic response of nonlinear systems under combined normal and Poisson white noise via path integral method

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ABSTRACT

In this paper the response in terms of probability density function of nonlinear systems under combined normal and Poisson white noise is considered. The problem is handled via a Path Integral Solution (PIS) that may be considered as a step-by-step solution technique in terms of probability density function. A nonlinear system under normal white noise, Poissonian white noise and under the superposition of normal and Poisson white noise is performed through PIS. The spectral counterpart of the PIS, ruling the evolution of the characteristic functions is also derived. It is shown that at the limit when the time step becomes an infinitesimal quantity an equation ruling the evolution of the probability density function of the response process of the nonlinear system in the presence of both normal and Poisson White Noise is provided.

1. Introduction

Normal and non-normal white noises are very popular stochastic processes and they have been used to model several types of physical phenomena, observed in physics, seismology, electrical engineering, economics and in some other research fields.

In particular in the engineering field very commonly a seismic excitation may be considered as a Gaussian process similarly to a wind excitation. Poisson processes are suitable for simulating the vehicular traffic on a bridge, or random ice impulsive loads on a ship. Then it will be very interesting having tools for knowing the most important features of the structural response under all these kinds of excitations considering also that they may act simultaneously.

For instance, if the input on the system is modelled as a normal White Noise the equation ruling the evolution of the probability density function (PDF) is usually referred to as the Fokker–Planck–Kolmogorov (FPK) equation. However, exact analytical expressions for the stationary response PDF of the FPK equation are known only for a very restrict class of non-linear systems [1–5].

For this reason the remaining problems must be treated by approximate procedures analytically or numerically [6–13]. Among the approximate methods, the Path Integral Solution (PIS) method is an effective tool giving the PDF of the response solution of the system at a given time instant when the PDF at an earlier close time instant is already known and then it may be seen as a step-by-step solution technique. Numerical solutions to implement the PIS based on interpolation schemes are available in literature [14–20]. It has been noted that, using a step-by-step PIS method, great accuracy at the tails of the PDF is evidenced and this result is very important for reliability analysis.

Many other random phenomena, like loads travelling on bridges and so on, are basically non-normal and most of them may be adequately modelled as Poissonian white noise. The equation ruling the evolution of the PDF is the Kolmogorov–Feller (KF) equation that is an integro-differential equation whose stationary solution is known only for certain types of input and non-linear forms of the system [21,22]. Solutions for this case by cell-to-cell path integration technique have been devised by [23–25], and in [26,27] provided an extension of PIS to Poissonian white noise. In a recent paper [28], PIS has been demonstrated to be a very efficient method for evaluating the PDF response of a system driven by normal and Poissonian white noise acting simultaneously.

This paper aims at evaluating the spectral counterpart of the Path Integral Solution for systems driven by normal and Poisson white noise acting simultaneously. Once the latter expression is found, then by making an inverse Fourier transform, the equation ruling the evolution of PDF for a nonlinear system enforced by superposition of normal and Poisson White Noise is introduced.

2. Path Integral Solution

Let the equation of motion of a half oscillator, driven by white noise, be given in the form:

$$\begin{cases} \dot{X}(t) = f(X,t) + W(t) \\ X(0) = X_0 \end{cases} \tag{1}$$

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where $f(X,t)$ is a deterministic nonlinear function of $X(t)$ and t; $W(t)$ is white noise (zeroth order memory Markov process) and X_0 is the relevant initial condition that may be either deterministic or a random variable (Gaussian or not Gaussian).

The Path Integral Solution (PIS) allows us to capture step-by-step the entire evolution of the response process in terms of PDF, starting from an assigned initial condition (deterministic or stochastic), to the steady state condition, that means if we suppose that the PDF of the response at the generic time instant (t) is already known, we may evaluate the PDF of the response at the time instant $(t+\tau)$ with τ small. In this hypothesis, because of the Markovianity of the input and of the response, the Chapman–Kolmogorov equation

$$p_X(x,t+\tau)=\int_D p_X(x,t+\tau|\bar{x},t)\,p_X(\bar{x},t)\mathrm{d}\bar{x} \tag{2}$$

holds true. Eq. (2) is valid for every value of τ, the only limitation is the Markovianity of the input, but for τ small Eq. (2) may be interpreted as a step-by-step method. For the numerical implementation of the PIS method a computational domain D has to be selected. It is convenient to select a symmetrical computational domain with a maximum size equal to x_1, i.e. $-x_1\le x\le x_1$. The size of the domain is identified by first running a MCS with a low number of samples. Then, dividing the domain in a number n_x of intervals, for each grid point, the path integral from Eq. (2) can be evaluated. By looking at this Eq. (2), it is apparent that the crucial point is to evaluate the kernel, where a conditional joint PDF is present. In recent papers [20,26–28], this conditional joint PDF has been determined in an original way as follows. From the whole trajectories of the response process $X(t)$ we take those starting from the box $(\bar{x}-\mathrm{d}\bar{x}\div\bar{x}+\mathrm{d}\bar{x})$ labelled $\bar{X}(\rho)$ (Fig. 1) that is, we find the unconditional PDF in τ of the following differential equation:

$$\begin{cases}\dot{\bar{X}}(\rho)=f(\bar{X},\rho)+W(t+\rho)\\ \bar{X}(0)=\bar{x}\end{cases} \tag{3}$$

being $\bar{x}$ a deterministic initial condition and $0\le\rho\le\tau$. The CPDF of Eq. (1) coincides with the unconditional PDF of Eq. (3) evaluated in τ, that is

$$p_X(x,t+\tau|\bar{x},t)=p_{\bar{X}}(x,\tau). \tag{4}$$

In Fig. 1 is shown the significance of the CPDF, that is the distribution of the stochastic process $\bar{X}(\rho)$ evaluated in $\rho=\tau$. These are the general features of PIS now the problem is to particularize the kernel and this will be dependent on the system and on the type of white noise.

3. Path Integral Solution for systems under normal white noise

Starting from the simplest case that is the nonlinear system enforced by a normal white noise labelled as $W_0(t)$ characterized by the correlation function:

$$E[W_0(t_1)W_0(t_2)]=q(t_1)\delta(t_1-t_2)=q(t_2)\delta(t_1-t_2) \tag{5}$$

$E[\bullet]$ being the ensemble average, $q(t)$ is the strength of the normal white noise (if $W_0(t)$ is stationary then $q(t)=q$) and $\delta(\bullet)$ the Dirac's delta function.

For such a system the equation of motion is Eq. (1) replacing $W(t)$ with $W_0(t)$

$$\begin{cases}\dot{X}(t)=f(X,t)+W_0(t)\\ X(0)=X_0\end{cases} \tag{6}$$

and the probability density function is the solution of Fokker–Planck–Kolmogorov (FPK) equation here reported

$$\frac{\partial}{\partial t}p_X(x)=-\frac{\partial}{\partial x}\left((f(x,t))\,p_X(x)\right)+\frac{q}{2}\frac{\partial^2 p_X(x)}{\partial x^2}. \tag{7}$$

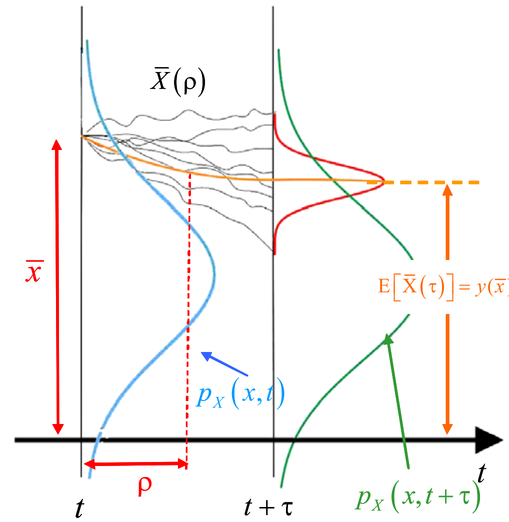

Fig. 1. Sample functions of $\bar{X}(\rho)$ and conditional PDF.

Eq. (7) is supplemented by the initial condition $p_X(x,0)=p_{X_0}$. For such an equation only the steady solution (if it exists) is known in the closed form solution. In order to overcome this problem the PIS will be used. Aiming at this the Chapman–Kolmogorov equation will be used and the conditional PDF is necessary. Indeed for this simple case the conditional probability density function (CPDF) $p_X(x,t+\tau|\bar{x},t)$ exists in the form provided by Risken [29]. In fact, invoking the so called short time Gaussian approximation, that is: although the system (1) is nonlinear, for τ small the CPDF $p_X(x,t+\tau|\bar{x},t)$ follows a Gaussian distribution, Risken provided the following CPDF

$$p_X(x,t+\tau|\bar{x},t)=\frac{1}{\sqrt{2\pi q\tau}}\exp\left(-\frac{(x-y(\bar{x}))^2}{2q\tau}\right) \tag{8}$$

where $E[X(t+\tau)]=\bar{x}+f(\bar{x},t)\tau=y(\bar{x})$. Moreover, as aforementioned, the conditional probability density function (CPDF) $p_X(x,t+\tau|\bar{x},t)$ present in the kernel of Eq. (2) may be obtained evaluating the unconditional PDF in τ of the following differential equation:

$$\begin{cases}\dot{\bar{X}}(\rho)=f(\bar{X},\rho)+W_0(t+\rho)\\ \bar{X}(0)=\bar{x}.\end{cases} \tag{9}$$

Evaluating in τ the mean and the variance of Eq. (9), in which all the quantities at the right hand side are calculated at the initial point it follows that

$$\begin{aligned}&E[\bar{X}(\tau)]=\bar{x}+f(\bar{x},t)\tau=y(\bar{x})\\&\sigma^2_{\bar{X}}(\tau)=q\tau\end{aligned} \tag{10}$$

and then

$$p_{\bar{X}}(x,\tau)=\frac{1}{\sqrt{2\pi q\tau}}\exp\left(-\frac{(x-y(\bar{x}))^2}{2q\tau}\right) \tag{11}$$

that totally coalesces with the expression (8) of Risken attesting the validity of the assertion in Eq. (4). The differential equations in terms of mean and mean square in τ are solved by linear approximation. Moreover, It has been shown that at the limit when $\tau\to 0$, the Chapman–Kolmogorov equation by using the kernel in (8) reverts to the Fokker–Planck equation [26,27].