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A stochastic collocation method for large classes of mechanical problems with uncertain parameters

Ph. Bressolette [a,*], M. Fogli [a], C. Chauvière [b]

[a] Laboratoire de Mécanique et Ingénieries, UBP & IFMA, 63174 Aubière Cedex, France

[b] Laboratoire de Mathématiques, UBP, 63177 Aubière Cedex, France

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ABSTRACT

The paper presents a self-contained and didactic approach to the stochastic collocation method. The method relies on the Lagrange polynomials and the Gauss quadrature rule. It is presented for large classes of mechanical problems, i.e. static problems, dynamic problems and spectral problems. After a general presentation of each of them, examples and results are provided. Numerical results show the high rate of convergence of the proposed method.

1. Introduction

The behaviour of most mechanical systems may be described by a vector equation whose nature (algebraic equation, differential equation, …) depends on the underlying mechanical problem to treat. Three important classes of problems can be distinguished: those of a static nature, those of a dynamic nature and those of a spectral nature. Numerical methods to treat these problems (Finite Elements Method, Discrete Elements Method, Spectral Elements Method, …) have now reached some degree of maturity to give accurate solutions through the adaptivity associated with efficient error estimators. However, the parameters of these equations are usually derived from experimental data and therefore are soiled by uncertainties. To accurately predict the behaviour of such systems, it becomes essential to take into account these uncertainties through a suitable probabilistic modelling. Numerical techniques are then required to quantify their probabilistic effects on the mechanical response.

These techniques may be classified in two major groups: sampling procedures, and non-sampling procedures. The best known sampling procedure is the Monte-Carlo method, or some of its refinements [1]. It consists in generating a large number of realisations of the random parameters from which a deterministic code gives the corresponding realisations of the random response of the system. These realisations are then statistically treated in order to estimate a few moments of the response. The main drawback of this approach is its slow rate of convergence. For example, the mean value typically converges as $1/\sqrt{N}$, where N is the number of realisations.

Moment, perturbation and spectral finite element stochastic methods [2] belong to non-sampling methods. Although they exhibit better rates of convergence than Monte-Carlo simulations, they present limitations that have prevented them from being widely used. In particular, most of them have an intrusive character that complicates their implementation on computers. Moreover, these methods are mostly efficient when the embedded mechanical model is mechanically linear. Furthermore, for the moment or perturbation methods, there is a high computational cost when the mechanical system exhibits a high degree of variability. The spectral stochastic finite element stochastic [3] method takes into account the random fields through truncated functional expansions using a finite number (generally small) of r.v.'s. This method has been applied with great success [4] and convergence studies [5]—both theoretical and numerical—have shown that it exhibits a fast convergence rate with increasing orders of expansion. For an extensive review, the reader might refer to [6].

In this paper, we do a self-contained presentation of a stochastic finite element method based on a stochastic collocation procedure [7]. Whereas the spectral element method uses orthogonal polynomials, the main feature of the proposed method lies in the

* Corresponding author. Tel.: +33 473 407 526; fax: +33 473 407 494.
E-mail address: philippe.bressolette@polytech.univ-bpclermont.fr (Ph. Bressolette).

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use of non-orthogonal Lagrange polynomials to approximate the nonlinear stochastic response, after having rewritten the problem in an appropriate Gaussian context. Moreover, it combines the fast convergence of the Galerkin methods with the non intrusive nature of the Monte-Carlo methods. The presentation has been conceived in a didactic spirit, with a view to showing the practical interest of the stochastic collocation method in random mechanics. That is why the technical aspects of this method have been emphasised over the mathematical aspects.

This paper is organised as follow: in the next section, we specify the classes of problems and the associated hypotheses for which the proposed method applies and we give typical examples of each class. Then, we present the general ideas used in the stochastic collocation method and we detail the application of the proposed method in the cases of static, dynamic and spectral problems. Section 4 illustrate the stochastic collocation method on some academic examples for which reference solutions exist (one of each class) followed by an engineering example. Finally, in Section 5 we draw the conclusions.

2. Classes of considered problems, associated hypotheses and typical examples

2.1. Classes of the considered problems

In this paper, we consider three classes of problems for which the stochastic collocation method is particularly well suited:

- Class 1: Static problems.
  Estimate the first few moments, and notably the mean $\mathbf{m_Y} = \mathbb{E}[\mathbf{Y}]$ and the correlation $\mathbf{R_Y} = \mathbb{E}[\mathbf{YY}^T]$, of the $\mathbb{R}^d$-valued random variable (r.v.) $\mathbf{Y} = (Y_1, \ldots, Y_d)^T$, such that
  $$\mathbf{Y} = \mathbf{f}(\mathbf{X}), \tag{1}$$
  where $\mathbf{X} = (X_1, \ldots, X_p)^T$ is a $\mathbb{R}^p$-valued r.v. with known probability distribution, $\mathbf{f}$ is a given nonlinear deterministic function from $\mathbb{R}^p$ into $\mathbb{R}^d$, p and d are two integers ≥ 1, and $\mathbb{E}[\cdot]$ is the mathematical expectation.
- Class 2: Dynamic problems.
  Estimate the first few moments, and in particular the mean function $\mathbf{m_Z}(t) = \mathbb{E}[\mathbf{Z}(t)]$, $t \in \mathbb{R}_+$, and the correlation function $\mathbf{R_Z}(s, t) = \mathbb{E}[\mathbf{Z}(s)\mathbf{Z}^T(t)]$, $(s, t) \in \mathbb{R}_+ \times \mathbb{R}_+$, of the $\mathbb{R}^d$-valued stochastic process $\mathbf{Z} = (\mathbf{Z}(t) = (Z_1(t), \ldots, Z_d(t))^T, t \in \mathbb{R}_+)$, such that, $\forall t \in \mathbb{R}_+$
  $$\mathbf{Z}(t) = \mathbf{g}(\mathbf{Q}(t)), \tag{2}$$
  where $\mathbf{g}$ is a given deterministic function from $\mathbb{R}^l$ into $\mathbb{R}^d$ and $\mathbf{Q} = (\mathbf{Q}(t) = (Q_1(t), \ldots, Q_l(t))^T, t \in \mathbb{R}_+)$ is a $\mathbb{R}^l$-valued stochastic process governed by the second order differential equation
  $$\begin{cases} \ddot{\mathbf{Q}}(t) + \mathbf{h}(\mathbf{Q}(t), \dot{\mathbf{Q}}(t), \boldsymbol{\xi}(t), t, \mathbf{V}) = 0, & t > 0 \\ \left(\mathbf{Q}^T(0), \dot{\mathbf{Q}}^T(0)\right)^T = \mathbf{H}_0(\boldsymbol{\Theta}_0) & \text{a.s.}, \end{cases} \tag{3}$$
  where $t \in \mathbb{R}_+$ is the time, $\ddot{\mathbf{Q}}$ and $\dot{\mathbf{Q}}$ are respectively the first and the second derivatives of $\mathbf{Q}$ with respect to t, $\mathbf{V} = (V_1, \ldots, V_m)^T$ is a $\mathbb{R}^m$-valued r.v. $\mathbf{H}_0$ is a given regular deterministic function from $\mathbb{R}^{2l}$ into $\mathbb{R}^{2l}$, $\boldsymbol{\Theta}_0 = (\Theta_{01}, \ldots, \Theta_{02l})^T$ is a $\mathbb{R}^{2l}$-valued r.v. independent of $\mathbf{V}$, $\boldsymbol{\xi}$ and $\mathbf{h}$ are given deterministic functions from $\mathbb{R}_+$ into $\mathbb{R}^l$ and from $\mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{R}_+ \times \mathbb{R}^m$ into $\mathbb{R}^l$ respectively, d, l, and m are three integers ≥ 1, $\mathbf{X} = (\mathbf{V}^T, \boldsymbol{\Theta}_0^T)^T$ is a $\mathbb{R}^p$-valued r.v. (with $p = m + 2l$), whose probability distribution is known, and a.s. is the abbreviation of "almost surely". Note that the solution process $\mathbf{Q}$ can be seen as a function of $\mathbf{X}$, i.e. $\mathbf{Q}(t) = \mathbf{r}(t, \mathbf{X})$, $\forall t \in \mathbb{R}_+$. Consequently, the process $\mathbf{Z}$ defined by Eq. (2) can also be seen as a function of t and $\mathbf{X}$:
  $$\mathbf{Z}(t) = \mathbf{G}(t, \mathbf{X}), \quad t \in \mathbb{R}_+. \tag{4}$$
- Class 3: Spectral problems.
  Estimate the second order statistics of the modal characteristics (eigenvalues, eigenvectors) of an undamped l-dimensional discrete linear dynamic system whose mass matrix $\mathcal{M} \in \mathbb{R}^{l\times l}$ and stiffness matrix $\mathcal{K} \in \mathbb{R}^{l\times l}$ depend on a $\mathbb{R}^p$-valued r.v. $\mathbf{X} = (X_1, \ldots, X_p)^T$ with a given probability distribution.

2.2. Hypotheses

To treat the previous problems, the following hypotheses will be assumed:

(H1) All the considered random quantities (i.e. the r.v.'s $\mathbf{X}$, $\mathbf{Y}$ and the stochastic processes $\mathbf{Z}$ and $\mathbf{Q}$) are defined on the same probability space $(\mathcal{A}, \mathcal{F}, P)$, where $\mathcal{A}$ is a sample space, $\mathcal{F}$ is a σ-algebra of parts of $\mathcal{A}$ and P is a probability measure on $\mathcal{F}$.

(H2) The p-dimensional r.v. $\mathbf{X} = (X_1, \ldots, X_p)^T$ follows a standard Gaussian distribution on $\mathbb{R}^p$. Hence, its components $X_1, \ldots, X_p$ are mutually independent and identically distributed according to a standard Gaussian distribution on $\mathbb{R}$. In these conditions, denoting by ψ_i the probability density function (pdf) of the component X_i of $\mathbf{X}$, such that

$$\psi_i(x_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}}, \quad x_i \in \mathbb{R}, \tag{5}$$

the pdf Ψ of $\mathbf{X}$ can be written

$$\Psi = \psi_1 \otimes \cdots \otimes \psi_p, \tag{6}$$

with, $\forall \mathbf{x} = (x_1, \ldots, x_p) \in \mathbb{R}^p$,

$$\Psi(\mathbf{x}) = (\psi_1 \otimes \cdots \otimes \psi_p)(\mathbf{x}) = \psi_1(x_1) \times \cdots \times \psi_p(x_p), \tag{7}$$

where $\otimes$ denotes the tensor product and $\times$ is the usual symbol of multiplication in $\mathbb{R}$.

(H3) The functions $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ are at least piecewise continuous, a property that ensures the validity of all the mathematical and numerical developments considered in the construction of the proposed method.

Remark. It is important to note that the hypothesis (H2) is not a loss of generality. Indeed, if $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_p)^T$ is a non-degenerate and continuous p-dimensional r.v. with a given pdf, then it is always possible to construct a regular transformation $\mathbf{T}$, with an inverse $\mathbf{T}^{-1}$, such that $\mathbf{T}^{-1}(\boldsymbol{\eta})$ is a p-dimensional standard Gaussian r.v., i.e. such that $\boldsymbol{\eta}$ can be written $\boldsymbol{\eta} = \mathbf{T}(\mathbf{X})$. Several techniques for constructing this transformation (in particular the well known techniques of Nataf, Hermite, Winterstein and Rosenblatt) can be found in references [8–10]. As a result, any problem expressed in terms of $\boldsymbol{\eta}$ can be expressed in terms of $\mathbf{X}$ using such a transformation.

2.3. Typical examples

The three classes of problems considered above represent a significant part of the set of structural reliability problems focused on the propagation of the uncertainties in numerical models. As an illustration, we give a typical example of problem for each class.

2.3.1. Example of static problem

This example consists of a mechanical system whose static linear behaviour is described by a l-degrees of freedom finite element model with a stiffness matrix $\mathcal{K} \in \mathbb{R}^{l\times l}$ (assumed to be regular), with the vector of nodal forces $\mathbf{e} \in \mathbb{R}^l$ and with the vector of nodal displacements $\mathbf{u} \in \mathbb{R}^l$. Let $\mathbf{y} \in \mathbb{R}^d$ be some observation of $\mathbf{u}$ (i.e. a vector of $\mathbb{R}^d$ containing stresses and/or strains and/or particular displacements…) linked to $\mathbf{u}$ via a continuous function